SUPPRESSION OF ULTRAVIOLET INFINITIES IN GRAVITY-MODIFIED QUANTUM FIELD THEORIES

by

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SUPPRESSION OF ULTRAVIOLET INFINITIES IN GRAVITY-MODIFIED QUANTUM FIELD THEORIES

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MAYASANDRA SUBRAHMANYA SRIRAM

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CERTIFICATE

Certified that the work presented in this thesis entitled, "Suppression of Ultraviolet Infinities in Gravity-Modified Quantum Field Theories" by M.S. Sri Ram has been carried out under my supervision and that this has not been submitted elsewhere for a degree.

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CONTENTS

SYNOPSIS

Thesis entitled, 'Suppression of Ultraviolet Infinities in Gravity-Modified Quantum Field Theories' submitted by MAYASANDRA SUBRAHMANYA SRIRAM in partial fulfilment of the requirements of the Ph.D. degree to the Department of Physics, Indian Institute of Technology, Kanpur.

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It is well known that a large class of nonpolynomial Lagrangians have an inbuilt cutoff in them. In the present work, ultraviolet infinities in quantum field theories are regularized by incorporating the gravitational interaction and applying nonpolynomial Lagrangian techniques to it. The calculations are done in a certain approximation to tensor gravity. The inverse of the gravitational coupling constant, κ, appears as an effective cutoff. This regularizing role of gravity is studied in some detail, in non-Abelian gauge theories and other renormalizable and nonrenormalizable field theories.

The thesis is divided into seven chapters. The first chapter is an introduction. In the second chapter we introduce nonpolynomial Lagrangian techniques. Ambiguities in the definition of the superpropagator and 'calculus of derivatives' for Lagrangians containing derivatives of fields are discussed.

In the third chapter, we show that by incorporating the couplings with the gravitational field, the ultraviolet infinities in quantum field theories are regularized, with the inverse of the gravitational coupling constant, κ, playing the role of a cutoff. An approximation which greatly simplifies the calculations is introduced and it is shown that it gives the same expressions for the renormalized amplitudes as the full tensor gravity in the lowest order in κ. Quantum electrodynamics is taken as an illustrative example. Next, it is shown that the regularizing effect is sufficiently strong to take care of the infinities of the socalled 'unrenormalizable' field theories also.

In the fourth chapter, we consider the regularization of theories with spontaneous symmetry-breaking through gravity, taking σ-model as an example. A special feature of this is the regularization of 'bubble' diagrams, in which an internal line closes in on itself. Vertex and self-energy corrections are calculated in the lowest order and it is shown that the counter-terms (finite in our case) are the same as those in the symmetric theory to $O(\ln \kappa)$. The correction to the vacuum expectation value of σ is computed and it is shown that the Goldstone theorem is verified only upto $O(\ln\kappa)$ and not to $O(\kappa^{0})$. The reason for this is traced.

In the fifth chapter, we consider the regularization of non-Abelian gauge theories. We compute the self-energy

and vertex corrections and the renormalization constants for pure Yang-Mills fields. It is seen that the result in gauge invariant only upto $O(\ln \kappa)$ and not to $O(\kappa^{0})$. This can be remedied by using a manifestly gauge covariant formalism for Yang-Mills fields with the currents defined carefully by the 'point-splitting' method. Next, we present the calculation of renormalization constants in Quantum Chromodynamics. Ward-identities are satisfied upto $O(\ln\kappa)$. In the last section, we briefly discuss the regularization of spontaneously broken gauge theories.

In the sixth chapter, we take up the problem of axial vector anomaly when fermions are present in a gauge theory, in our framework. In the gravity-modified theory also there is an anomalous term in the axial vector Ward identity, which is the same as the standard one to $O(\kappa^{0})$. This we show, by explicitly evaluating the contribution of the triangle graph to the vector-vector-axial vector vertex in gravity-modified quantum electrodynamics. Defining the currents carefully using the point-splitting method and equations of motion, we get the same answer. The latter method can be extended to non-Abelian gauge theories also.

The final chapter is devoted to a few concluding remarks.

Additional Note to Synopsis

After the Synopsis was submitted, we found that in the gravity-modified σ-model that we have considered, the lowest order vertex and self-energy corrections are consistent with the global symmetry of the theory to $O(\kappa^0)$. Also, Goldstone theorem and PCAC are validated to $O(\kappa^0)$ and not only to $O(\log \kappa)$ as stated in the Synopsis.

CHAPTER I

INTRODUCTION

It is well known that there are ultraviolet divergences in quantum field theories [1]. It has been shown that these divergences persist in the exact solutions of polynomial Lagrangian field theories in two and three space-time dimensions [2]. If this conclusion may be extrapolated to the physical four-dimensional space-time, it appears that the fault does not lie in the perturbation expansion, but in the nature of the interactions. Indeed, it has been shown for a variety of nonpolynomial Lagrangian theories that they possess remarkable convergence properties [3-6]. Nonpolynomial Lagrangians occur naturally in chiral theories (with the pseudoscalar mesons in a nonlinear representation of SU(2) x SU(2) or SU(3) x SU(3)), intermediate-vector-boson mediated weak interactions, and in fact in all theories if we include gravitational field which couples to all fields and is intrinsically nonpolynomial [4-6].

It has been conjectured in the past that the universal nonlinear coupling of gravitation may cure the infinities in quantum field theories [7]. Using nonpolynomial Lagrangian field theory techniques Isham, Salam and Strathdee have shown that the inverse of the gravitational coupling constant, κ, acts as an inbuilt cutoff in gravity-modified quantum electrodynamics [8,9].

This infinity-suppression mechanism is expected to work for a general quantum field theory, when the couplings with the gravitational field are included. But the complications of tensor gravity make it very difficult to put it in practice. Tulsi Dass and Radhey Shyam [10,11] considered quantum electrodynamics modified by including the exponential coupling of a massless scalar field (which is essentially scalar gravity) in a conformal invariant manner [12] and showed that the ultraviolet divergences are absent from the theory to all orders in perturbation. This method can be extended to all Lagrangians which have terms with scale dimension different from -4.

This procedure, however, does not work for theories which are already conformal-invariant, i.e., when all the terms in the Lagrangian have scale dimension -4. An example is pure Yang-Mills theory. By ascribing a scale dimension to the field variable different from the canonical dimension, it is possible to include exponential couplings of the scalar field in the Lagrangian. But this is adhoc and does not appear convincing. Hence, we have to fall back on the gravitational field which couples to all the fields.

In the present work, we study the regularizing role of gravitational couplings when they are included in various Lagrangians of physical interest, namely theories with

spontaneous symmetry breaking, non-Abelian gauge theories and theories involving axial-vector currents. As the graviton-superpropagators have a complicated structure [8,13] the calculations are done in a suitable approximation to the tensor-gravity. This is the same approximation that is utilized in [9] and involves neglecting κ-dependent factors in $g^{\mu\nu}$, i.e., replacing these by $\eta^{\mu\nu}$ and retaining them in $\det g$ only. Also gravitational self-interactions are not taken into account. This approximate version gives the same results for the renormalized amplitudes as the full tensor gravity to $O(\kappa^0)$. As expected, the inverse of the gravitational constant plays precisely the role of a cutoff. Due attention has been paid to the problem of maintaining the various symmetry relations in perturbation theory with this regularization.

The thesis is planned as follows: in the second chapter, we introduce nonpolynomial Lagrangian techniques including a discussion of ambiguities and 'calculus of derivatives' for derivative-containing Lagrangians. In the third chapter, we give the details of the approximation we make when gravitational interactions are incorporated in a Lagrangian and give the justification for it. Quantum electrodynamics is taken as an illustrative example. It is shown that the regularizing mechanism works for nonrenormalizable field theories also.

The fourth chapter deals with the regularization of theories with spontaneous symmetry-breaking with σ-model as an example. A special feature of this is the regularization of the 'bubble' diagrams. Vertex and self-energy corrections and the vacuum expectation value of σ are calculated in the lowest order. Goldstone theorem and PCAC are discussed.

In the fifth chapter, we consider the regularization of non-Abelian gauge theories. Self-energy and vertex corrections for pure Yang-Mills theory as well as quantum chromodynamics are evaluated. Gauge invariance is discussed. Spontaneously broken gauge theories are also considered.

In the sixth chapter, the status of axial vector anomaly in a gravity-modified gauge theory is examined by the explicit evaluation of the triangle graph as well as through the equations of motion.

In the final chapter we make a few concluding remarks.

CHAPTER II

NONPOLYNOMIAL LAGRANGIANS

In this chapter, we introduce nonpolynomial Lagrangian techniques using some simple illustrative models. Ambiguities in the definition of the superpropagator and 'calculus of derivatives' for Lagrangians containing derivatives are discussed.

2.1 Nonpolynomial Lagrangians Without Derivatives

Consider the simplest nonpolynomial Lagrangian [14,15],

$$L = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + G : e^{\lambda\phi} : \tag{2.1}$$

Here onwards, double dots denote normal ordering. The interaction-part of the Lagrangian is,

$$L_{int} = G : e^{\lambda\phi} : = G \sum_{n=0}^{\infty} \frac{\lambda^n : \phi^n :}{n!} \tag{2.2}$$

It is well known that the polynomial interaction $:\phi^n:$ has ultraviolet divergence which cannot be removed by renormalization if $n > 4$. Thus the expansion (2.2), if treated term by term would give intractably divergent results. But by expanding in powers of G and resummation of the series in λ, we get an entirely different result.

The S-matrix arising from (2.1) is:

$$\begin{aligned} S &= T\left[\exp\{ i \int L_{int}\, d^4x \}\right] \\ &= \sum_{N=0}^{\infty} \frac{i^N}{N!} G^N \int dx_1 \ldots dx_N\, T\{ : e^{\lambda\phi_1} \ldots e^{\lambda\phi_N} : \} \end{aligned} \tag{2.3}$$

Here $\phi_i = \phi(x_i)$ and T denotes time ordering. It can be shown that:

$$T\{\prod_{i=1}^{N} :e^{\lambda\phi_i}:\} = \prod_{\substack{i,j=1 \\ i<j}}^{N} e^{\lambda^2 D_{ij}} :\prod_{i=1}^{N} e^{\lambda\phi_i}: \tag{2.4}$$

where $D_{ij} = D(x_i - x_j) = (0\,|\,T(\phi(x_i)\,\phi(x_j))|\,0)$

$$= -\frac{1}{4\pi^2}\,\frac{1}{(x_i - x_j)^2} \tag{2.5}$$

is the propagator for a massless scalar field.

The N^{th} order contribution to the S-matrix can be described in graphical form, in the coordinate space as follows. The graph consists of N vertices labelled $x_1, \ldots x_N$. The i^{th} of these vertices can have m_i external particles associated with it, the corresponding factor being $\lambda^{m_i}/m_i!$, as can be seen from eqn.(2.4). m_i ranges from 0 to ∞. The line joining i^{th} and j^{th} vertices corresponds to a factor $e^{\lambda^2 D_{ij}}$. This corresponds to the total sum of exchange of one, two, three, ... scalar particles and is called the superpropagator. Finally, there is an overall factor of $(iG)^N/N!$ for the graph, which may now be called a "supergraph". There is only one supergraph for each order of G (apart from permutations of the external particles). Similar rules can can be obtained for a supergraph in an arbitrary nonderivative nonpolynomial Lagrangian.

Consider a supergraph in second order in G, shown in Fig. 1.

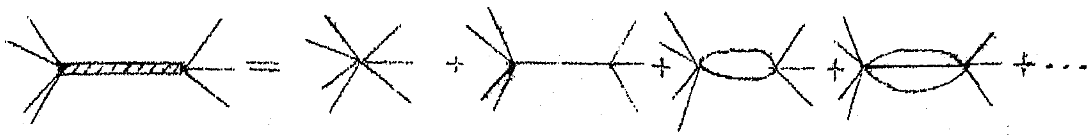

Fig.1 Second Order Supergraph. Double line with slashes corresponds to the superpropagator.

Here, only a single superpropagator is involved:

$$e^{\lambda^2 D(x)} = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{n!} D^n \qquad (2.6)$$

One can write a Sommerfeld-Watson transform for this:

$$e^{\lambda^2 D(x)} = \frac{1}{2\pi i} \int_{C_0} dz\, \Gamma(-z)(-\lambda^2)^z (D(x))^z \qquad (2.7)$$

where C_0 is a contour enclosing the positive real axis in the clock-wise sense. C_0 may be opened out to be contour C_0' running parallel to the imaginary axis with $-1 < \mathrm{Re}\, z < 0$. To evaluate the Fourier transform of the superpropagator, we use the Gelfand-Shilov formula [16]:

$$(D(x))^z = \frac{-i\,\Gamma(2-z)}{16\pi^2\Gamma(z)} \int \frac{d^4q}{(2\pi)^4} e^{-iq.x} \left(\frac{-q^2}{16\pi^2}\right)^{z-2} \qquad (2.8)$$

$$(0 < \mathrm{Re}\, z < 2)$$

[This holds for arbitrary z also, if we consider $(q^2)^{z-2}$ as a distribution which has poles at $z = 0, -1, -2, \ldots$ [16]].

Using this, we have for the Fourier transform of the superpropagator,

$$\tilde{\mathcal{D}}(q,\lambda) = \int e^{iq.x} e^{\lambda^2 D(x)} dx = \frac{-16\pi^2 i}{2\pi i} \int_{C_0'} \frac{dz\,\Gamma(-z)\Gamma(2-z)}{\Gamma(z)(q^2)^2} \left(\frac{\lambda^2 q^2}{16\pi^2}\right)^z \qquad (2.9)$$

We can fold back the contour C_0' to C_0 now. The integrand has simple poles at $z = 0$ and 1 and double poles at $z=2,3,\ldots$ Evaluating the residues, we find [14]:*

$$\widetilde{\mathcal{D}}(q,\lambda) = (2\pi)^4\ \delta^4(q) + \frac{i\lambda^2}{q^2} + \frac{16\pi^2 i}{(q^2)^2}\sum_{n=2}^{\infty}\Big[\Big(\frac{\lambda^2 q^2}{16\pi^2}\Big)^n$$
$$(\Gamma(n-1)\,\Gamma(n)\,\Gamma(n+1))^{-1}\Big\{\log\Big(\frac{\lambda^2 q^2}{16\pi^2}\Big) - \Psi(n-1) - \Psi(n) - \Psi(n+1)\Big\}\Big] \tag{2.10}$$

where $\Psi(n) = \Gamma'(n)/\Gamma(n)$.

All the amplitudes in this theory are just proportional to $\widetilde{\mathcal{D}}(q,\lambda)$ (q being a combination of external momenta) in the order G^2. Hence they are all finite. It is to be noted that the superpropagator is non-analytic in the coupling constant. [There are terms proportional to $\log(\lambda^2)$].

For higher order supergraphs the procedure for calculation would be as follows: (i) Sommerfeld-Watson transforms in eqn.(2.7) are written for each superpropagator with z replaced by z_{ij} for the superpropagator between i^{th} and j^{th} vertices, (ii) the contours are stretched to lie parallel to the imaginary axis with $-1 < \mathrm{Re}\, z_{ij} < 0$, (iii) Fourier transforms are performed using the Gelfand-Shilov formula, (iv) integrations over the momenta associated with the superpropagators are performed and (v) the contours are folded back to the original position and residues of the poles in z_{ij} variables are evaluated. Some finer modifications are

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* There are some errors in the expression for $\widetilde{\mathcal{D}}(q,\lambda)$ in Ref. [14].

necessary to ensure that the procedure is valid mathematically at each stage. In this manner, Taylor [15] has shown that the theory is finite, unitary and causal to all orders in the major coupling constant G.

Ambiguities:

The superpropagator in eqn.(2.6) and eqn.(2.7) has ambiguities. In the Sommerfeld-Watson transform in eqn.(2.7), we could have replaced $\Gamma(-z)$ by $\Gamma(-z)+a(z)$ where $a(z)$ does not have any poles on the real axis and vanishes sufficiently fast as $|z|\to\infty$. This is related to the ambiguity in the definition of $D(x)^n$ itself. $D(x)^n$ has ambiguities of the form:

$$\sum_{r=0}^{n-2} C_r(\partial^2)^r \delta^4(x)$$

In case of $e^{\lambda^2 D}$ the ambiguities are of the form

$$\sum_{n=0}^{\infty} a_n(\partial^2)^n \delta^4(x)$$

For the Fourier transform of the superpropagator, these terms would give additional contributions of the form

$$\sum_{n=0}^{\infty} a_n(k^2)^n$$

Now, the superpropagator in eqn.(2.10) has the property [14] that:

$$\mathrm{Re}(i\widetilde{\mathcal{F}}(q,\lambda)) \to 0 \quad \text{as } q^2 \to \infty. \tag{2.11}$$

The ambiguity terms will destroy this property. Hence, if we impose the condition (2.11), we should have $a_n = 0$ or $a(z) = 0$. This is the Lehmann-Pohlmayer ansatz [17]. It has been shown that this ansatz removes the ambiguities in the third order perturbation also [18].

Bollini and Giambiagi [19] define the Fourier transform of the distributions of the type $(D(x))^{\alpha} \exp(\lambda^2 D(x))$ by considering analytic interpolating functions relating $e^{\lambda^2 D(x)}$ in Minkowski space and the corresponding object in Euclidean space which is a well-behaved distribution. This approach ensures that $e^{\lambda^2 D(x)}$ and all its derivatives vanish at the origin. In this approach there are no ambiguities and in fact, for $\tilde{\mathcal{D}}(q, \lambda)$ they get an answer identical to the expression on the R.H.S. of eqn. (2.10); i.e., a_n are uniquely fixed at the value zero. This gives a mathematical justification for the physical 'minimal singularity' ansatz.

Rational Lagrangians:

The Lagrangian in eqn.(2.1) satisfies the Jaffe-criterion for localizability [20], namely the two-point spectral function $\rho(p^2)$ grows slower than $\exp\{|p^2|^{\frac{1}{2}}\}$ as $p^2 \to \infty$. In general, for a Lagrangian with

$$L_{int}(\phi) = \sum_{n=0}^{\infty} \frac{v(n)}{n!} : \phi^n : ,$$

the theory is localizable if $|v(n)| < A^n n^{\sigma n}$ with $0 < \sigma < \frac{1}{2}$ where A is a constant [21]. This condition is not satisfied for a rational Lagrangian like

$$L_{int}(\phi) = :(1-\lambda\phi(x))^{-1}: \qquad (2.12)$$

For this, the superpropagator is

$$F(x) = (0\,|T\{:(1-\lambda\phi(x))^{-1}:\ :(1-\lambda\phi(0))^{-1}:\}|\,0)$$

$$= \sum_{n=0}^{\infty} \lambda^{2n}\, n!\, D^n(x) \qquad (2.13)$$

This series has a zero radius of convergence in D(x). A formal sum may be defined by adopting the Borel summation procedure; in eqn. (2.13) use the relation

$$n! = \int_0^\infty e^{-t}\, t^n\, dt$$

and interchange the summation over n and integral over t to give

$$F(x) = \int_0^\infty \frac{e^{-t}\, dt}{1-\lambda^2 t\, D(x)} \qquad (2.14)$$

As $x^2 \to 0$, the integral vanishes like x^2. Hence, we would have finite amplitudes in this case also. However, for space-like x^2 the integrand has a pole in t at $t = \frac{1}{\lambda^2 D} = \frac{-4\pi^2 x^2}{\lambda^2}$. Then we would have to specify how to integrate over the pole. This ambiguity is referred to as the 'Borel ambiguity' [4]. The prescription of taking the principal value [22] ensures reality of the superpropagator in the space-like region but this may give rise to some unphysical results [23]. It is to be noted that localizable Lagrangian like the exponential Lagrangian do not have the Borel ambiguity.

Hence an exponential parametrization for the gravitational field is localizable and vastly superior to a rational parametrization.

2.2 Nonpolynomial Lagrangians Containing Derivatives

Derivative containing Lagrangians constitute the majority of the physically interesting cases. As a simple example, consider

$$L = :\tfrac{1}{2}\partial_\mu\chi\partial^\mu\chi: + :\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi\ e^{-\lambda\chi}: \qquad (2.15)$$

where ϕ and χ are massless scalar fields. We would have for the interaction part of Lagrangian,

$$L_{int} = :\tfrac{1}{2}(e^{-\lambda\chi} - 1)\ \partial_\mu\phi\partial^\mu\phi: \qquad (2.16)$$

In fact, this is the interaction part of the Lagrangian for a massless scalar field without self-interactions coupled to gravity, in the approximation to be discussed in Chapter III. Then in the second order, the self-energy $\Sigma(x)$ of the scalar field ϕ is given by

$$\Sigma\ (x) = +i\partial^\mu\partial^\nu\ [(\partial_\mu\partial_\nu D(x)) \sum_{n=1}^{\infty} \frac{\lambda^{2n}}{n!} D(x)^n] \qquad (2.17)$$

Using the Sommerfeld-Watson transform for the series on the R.H.S. of eqn. (2.17):

$$\Sigma(x) = \frac{1}{2\pi i} \int_{C_1} dz\ \Gamma(-z)(-\lambda^2)^z\partial^\mu\partial^\nu\ [(\partial_\mu\partial_\nu\ D)D^z] \qquad (2.18)$$

where C_1 is a contour enclosing the positive real axis from 1 to ∞ in the clock-wise sense. We see that we need to define product of distributions of the form

$$\partial_\mu \partial_\nu D \times D^z$$

Salam and co-workers[5] have proposed the following formulae which one gets after naive manipulations and which conserve the number of derivatives for product of distributions containing derivatives. These are together termed as 'calculus of derivatives' :

$$D^{z_1} \times \partial_\mu D^{z_2} = \frac{z_2}{(z_1+z_2)} \partial_\mu (D^{z_1+z_2}) \tag{2.19}$$

$$\partial_\mu D^{z_1} \times \partial_\nu D^{z_2} = \frac{z_1 z_2}{(z_1+z_2)(z_1+z_2+1)} \{\partial_\mu \partial_\nu + \frac{\eta_{\mu\nu}\partial^2}{2(z_1+z_2-1)}\} D^{z_1+z_2} \tag{2.20}$$

$$D^{z_1} \times \partial_\mu \partial_\nu D^{z_2} = \frac{z_2}{(z_1+z_2)(z_1+z_2+1)} \{(1+z_2)\, \partial_\mu \partial_\nu - \frac{z_1}{2(z_1+z_2-1)} \eta_{\mu\nu}\partial^2\} D^{z_1+z_2} \tag{2.21}$$

(except when $z_1 = 0$ and $z_2 = 1$)

Bollini and Giambiagi [19] have given justification for these when one of the distributions in the L.H.S. of eqns.(2.19 - 2.21) is replaced by $e^{\lambda^2 D(x)}$, using the method mentioned previously.

Using the relation (2.21), we have

$$\Sigma (x) = \frac{2i}{2\pi i} \int_{C_1} dz\, \Gamma(-z)(-\lambda^2)^z \frac{\partial^\mu \partial^\nu}{(z+1)(z+2)} [(\partial_\mu \partial_\nu - \tfrac{1}{4}\eta_{\mu\nu}\partial^2) D^{z+1}]$$

$$= \frac{3i}{2} \frac{1}{2\pi i} \int_{C_1} dz\, \Gamma(-z)(-\lambda^2)^z \frac{1}{(z+1)(z+2)} [(\partial^2)^2 D^{z+1}] \tag{2.22}$$

Taking the Fourier transform after bending the contour such that it lies parallel to the imaginary axis with $0 < \mathrm{Re}\, z < 1$ and folding it back, we get

$$\Sigma (k) = \int \Sigma(x)\, e^{-ik.x}\, dx$$

$$= \frac{3}{2}(k^2) \frac{1}{2\pi i} \int_{C_1} dz \frac{\Gamma(-z)\,\Gamma(1-z)}{\Gamma(z+1)(z+1)(z+2)} \left(\frac{\lambda^2 k^2}{16\pi^2}\right)^z \tag{2.23}$$

Evaluating the residue of the double-pole at $z=1$, we get

$$\Sigma(k) = \frac{\lambda^2 (k^2)^2}{64\,\pi^2} [\log(\frac{\lambda^2 k^2}{16\pi^2}) - 1 - 2\Psi(1) - \Psi(4)] + O(\lambda^4 \log(\lambda^2))$$

The use of 'calculus of derivatives' greatly simplifies the calculation.

The use of calculus of derivatives is particularly relevant in gauge theories. It is expected that the use of these formulae would resolve the gauge-invariance problem in these theories [5,24]. In a general gauge in gauge theories, we use the following additional formulae which one gets after similar manipulations.

$$D^{z} \times \frac{\partial_{\mu}\partial_{\nu} D}{\partial^{2}} = \frac{1}{2}\left[\frac{z}{z+1} \eta_{\mu\nu} - \frac{2(z-1)}{(z+1)} \frac{\partial_{\mu}\partial_{\nu}}{\partial^{2}}\right] D^{z+1} \tag{2.24}$$

$$\partial^{\nu}\left[\partial^{\rho} D \times \frac{\partial^{\mu}\partial_{\rho} D}{\partial^{2}} \times D^{z}\right] = -\frac{1}{2(z+2)} \partial^{\mu}\partial^{\nu} D^{z+2} \tag{2.25}$$

$$\partial^{\rho}\partial^{\nu} D \times \frac{\partial^{\mu}\partial_{\rho} D}{\partial^{2}} \times D^{z} = -\frac{1}{2(z+2)(z+3)}\left[\partial^{\mu}\partial^{\nu} + \frac{(z+4)}{2(z+1)} \eta^{\mu\nu}\partial^{2}\right] D^{z+2} \tag{2.26}$$

$$\partial^{\lambda}\partial^{\rho}\left[\frac{\partial_{\lambda}\partial_{\rho} D}{\partial^{2}} \times \frac{\partial^{\mu}\partial^{\nu} D}{\partial^{2}} \times D^{z} - \frac{\partial^{\mu}\partial_{\lambda} D}{\partial^{2}} \times \frac{\partial^{\nu}\partial_{\rho} D}{\partial^{2}} \times D^{z}\right]$$

$$= \frac{1}{4} \frac{z}{(z+2)}\left[\partial^{\mu}\partial^{\nu} - \eta^{\mu\nu}\partial^{2}\right] D^{z+2} \tag{2.27}$$

By these relations we mean that the Fourier transforms of both sides are equal. For example, consider the Fourier transform of L.H.S. of eqn. (2.24):

$$F\left[D^{z} \times \frac{\partial_{\mu}\partial_{\nu} D}{\partial^{2}}\right](k) = \frac{1}{(2\pi)^{4}} \int d^{4}p \, F\left(\frac{\partial_{\mu}\partial_{\nu} D}{\partial^{2}}\right)(p) \, F(D^{z})(k-p)$$

$$= \frac{1}{(2\pi)^{4}} \frac{\Gamma(2-z)}{\Gamma(z)} \frac{(-16\pi^{2})^{2-z}}{(16\pi^{2})} \int d^{4}p \, \frac{p_{\mu} p_{\nu}}{(p^{2})^{2}} \left[(k-p)^{2}\right]^{z-2} \tag{2.28}$$

On evaluating integrals we get the Fourier transform of the R.H.S. of eqn. (2.24):

$$F\left[D^{z} \times \frac{\partial_{\mu}\partial_{\nu} D}{\partial^{2}}\right](k) = \frac{1}{2}\left[\frac{z}{z+1}\eta_{\mu\nu} - \frac{2(z-1)}{(z+1)} \frac{k_{\mu}k_{\nu}}{k^{2}}\right] \frac{(-i)\Gamma(1-z)}{16\pi^{2}\Gamma(z+1)}\left(\frac{-k^{2}}{16\pi^{2}}\right)^{z-1}$$

$$= F\left[\frac{1}{2}\left\{ \frac{z}{z+1}\eta_{\mu\nu} - \frac{2(z-1)}{(z+1)} \frac{\partial_{\mu}\partial_{\nu}}{\partial^{2}}\right\} D^{z+1}\right](k)$$

The relations (2.24 - 2.27) are not needed for calculations in the Feynman gauge.

The relations (2.19-2.21) and (2.24-2.27) are useful only when massless propagators are involved. We do not know any generalization of these to include massive propagators also.

CHAPTER III

REGULARIZATION SCHEME AND APPROXIMATION

In this chapter, we show that by incorporating the couplings with the gravitational field, the ultraviolet infinities in quantum field theories are regularized with the inverse of the gravitational coupling constant, κ playing the role of a cutoff. An approximation to tensor gravity mentioned in reference [9] is introduced and it is shown that it gives the same expressions for the renormalized amplitudes as the full tensor gravity to $O(\kappa^{0})$. Quantum electrodynamics is taken as an illustrative example. Next, it is shown that the regularizing effect is sufficiently strong to take care of the infinities of even the 'unrenormalizable field theories'.

3.1 Gravity-Modified Quantum Field Theories

As is well known [25,8], when couplings with the gravitational field are introduced in a Poincare invariant Lagrangian, the following modifications are to be made:

(i) Ordinary derivatives of fields are to be replaced by the covariant derivatives employing the coefficients of affine connection.

(ii) The Dirac bilinears, which are tensors with respect to the Lorentz group are to be converted into tensors with respect to the general coordinate transformations with the help of the **vierbein** fields, $L^{a\mu} = L^{\mu a}$ related to the metric tensor $g^{\mu\nu}$ by (η^{ab} is the Minkowski metric):

$$g^{\mu\nu} = L^{\mu a}\,\eta_{ab}\,L^{b\nu} \tag{3.1}$$

(iii) All contractions of tensor indices are to be done with the Riemannian metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ instead of the Minkowski metrics $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$.

(iv) Multiply each term in the Lagrangian with an appropriate power of $\sqrt{(-g)}$ to make it a scalar density, where $g = \det(g_{\mu\nu})$.

(v) Add the Einstein Lagrangian for the gravitational field

$$L_{grav} = \frac{1}{\kappa^2}\sqrt{-g}\,R \tag{3.2}$$

where κ is the gravitational coupling constant and R is the curvature.

(vi) A gauge-fixing term for the gravitational field.

For example, the gravity-modified Lagrangian for quantum electrodynamics (QED) is [8]:

$$L = \sqrt{(-g)}[\tfrac{i}{2}(\bar{\psi}\gamma_a\psi_{;\mu} - \bar{\psi}_{;\mu}\gamma_a\psi)L^{\mu a} - m\bar{\psi}\psi + e\bar{\psi}\gamma_a\psi L^{\mu a}A_\mu$$

$$- \tfrac{1}{4}F_{\rho\sigma}F_{\mu\nu}g^{\mu\rho}g^{\nu\sigma}] + L_{grav} + L_{gauge} \tag{3.3}$$

where

$$\psi_{;\mu} = \partial_\mu\psi - \tfrac{1}{4}\,i\,B_{\mu ab}\,\sigma^{ab}\psi$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$B_{\mu ab} = \tfrac{1}{2}(L_a^{\ \nu}\partial_\mu L_{\nu b} - L_b^\nu\partial_\mu L_{\nu a}) - \tfrac{1}{2}(L_a^\nu\,\partial_\nu L_{\mu b} - L_b^\nu\,\partial_\nu L_{\mu a})$$

$$- \tfrac{1}{2}L_{\mu c}(\partial_\lambda L_\nu^c - \partial_\nu L_\lambda^c)L_a^\lambda L_b^\nu \tag{3.4}$$

$$\sigma^{ab} = \tfrac{i}{4}[\gamma^a, \gamma^b]$$

Here $L_{\mu a} = g_{\mu\nu}\,\eta_{ab}\,L^{\nu b}$. The contractions of vierbeins in eqn.(3.4) are done with $\eta^{\mu\nu}$.

We now introduce the exponential parametrization [9]

$$L^{\mu a} = [\exp(\tfrac{\kappa}{2}\phi)]^{\mu a} \tag{3.5}$$

which implies

$$g = -e^{-\kappa \mathrm{Tr}\,\phi} \tag{3.6}$$

Here ϕ is the symmetric 4x4 matrix field of gravitons. In the Fock-deDonder gauge for gravity [26],

$$(0|T(\phi^{\mu a}(x)\,\phi^{\nu b}(0))|0) = \tfrac{1}{2}(\eta^{\mu\nu}\,\eta^{ab} + \eta^{\mu b}\,\eta^{\nu a} - \eta^{\mu a}\eta^{\nu b})D(x) \tag{3.7}$$

where D(x) denotes the zero-mass causal propagator, given by

$$D(x) = -\frac{1}{4\pi^2 x^2}$$

The Lagrangian (3.3) can now be written as

$$L = L_o + L_1 + L_2 + L_3 + L_{grav} + L_{gauge} \tag{3.8}$$

where

$$L_o = \tfrac{i}{2}\eta^{\mu a}(\bar{\psi}\gamma_a\,\partial_\mu\psi - \partial_\mu\bar{\psi}\,\gamma_a\psi) - m\bar{\psi}\psi - \tfrac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}F_{\mu\nu}\,F_{\rho\sigma}$$

$$L_1 = e\,\sqrt{(-g)}\,\bar{\psi}\gamma_a\psi\;A_\mu\;L^{a\mu}$$

$$L_2 = \tfrac{i}{2}(\sqrt{(-g)}L^{\mu a} - \eta^{\mu a})(\bar{\psi}\gamma_a\,\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma_a\psi) - (\sqrt{(-g)}-1)m\,\bar{\psi}\psi \tag{3.9}$$

$$-\tfrac{1}{4}(\sqrt{(-g)}\,g^{\mu\rho}g^{\nu\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})F_{\rho\sigma}F_{\mu\nu}$$

$$L_3 = \frac{\sqrt{(-g)}}{4}\,L^{\mu a}\,\bar{\psi}\gamma_a\,B_{\mu bc}\,\sigma^{bc}\,\psi\,.$$

It should be noted that the Lagrangians L_2, L_3 and the interaction part of L_{grav} are of order κ. Hence these terms

are generally expected to contribute terms of order κ or higher in a given amplitude, although caution should be exercised in drawing such a conclusion in a given case. [10].

We make an approximation to this Lagrangian, which is the same as the one mentioned in the work of Isham, et al. [9]. This consists in ignoring the gravitational self-interactions and replacing $L^{a\mu}$ by $\eta^{a\mu}$ in the rest of the Lagrangian. We now have, for the interaction part of the Lagrangian:

$$L_1' = L_1'' + L_{K.M.} \tag{3.10}$$

where $$L_1'' = e\,\bar{\psi}\gamma_a\psi\, A_\mu\, \eta^{a\mu}\, e^{-\kappa\chi} \tag{3.11}$$

and $$L_{K.M.} = (e^{-\kappa\chi} - 1)\,[\tfrac{i}{2}(\bar{\psi}\,\gamma_a\,\partial_\mu\psi - \partial_\mu\bar{\psi}\,\gamma_a\psi)\,\eta^{a\mu} - m\bar{\psi}\,\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\,] \tag{3.12}$$

Here we have put $\chi = \frac{1}{2}\mathrm{Tr}\,\phi$. From (3.7)

$$(0\,|\,T(\chi(x)\,\chi(o))|\,0) = -D(x)$$

so that

$$(0|\,T(:e^{-\kappa\chi(x)}:\; :e^{-\kappa\chi(o)}:)|\,0) = e^{-\kappa^2 D(x)} \tag{3.13}$$

L_1' is the same as $L_{e.m.}$ of Tulsi Dass and Radhey Shyam [10,11] with $f = -\kappa$. $L_{K.M.}$ is the kinetic energy modification term characteristic of a large class of non-polynomial Lagrangians [4,5]. We have included this term as it is crucial in the regularization of the bubble diagrams in theories in the spontaneous symmetry-breaking and gauge invariance in gauge theories, considered in the later chapters, though it is formally of order κ.

We assume that the gravitational part of the Lagrangian is also normal-ordered, i.e. we replace $e^{-\kappa\chi}$ by $:e^{-\kappa\chi}:$. It is known that naive normal ordering destroys the gauge properties of a theory like gravitation. But together with the 'calculus of derivatives' this is expected to give gauge-invariant results[5,24].

$$e^{-\kappa\chi} = e^{\left(\frac{D(0)}{2}\ \frac{\partial^2}{\partial\chi^2}\right)} :e^{-\kappa\chi}: \quad ,$$

So in effect we have renormalized D(0) to zero [4, 5].

We will now show that L_1 and L_1'' yield the same renormalization constants (for electron and photon wave function renormalization constants and the vertex renormalization) in lowest order in e and to order log κ (but not in the order κ^0). Also the renormalized amplitudes (to order κ^0) in the gravity-modified theory, with or without the above-mentioned approximation will be the same as in conventional QED. This will also make it clear why it should be a general feature. The effects of the other terms in the Lagrangian will be discussed later. We do not hope to keep the level of mathematical rigour of, for example Ref.[15]; however, we believe the arguments will be valid in a rigorous formulation also.

From (3.1), it is clear that the following graviton superpropagator enters in the calculations of self-energies and vertex corrections:

$$\mathcal{D}^{\mu a,\nu b}(x) = (0|T(\frac{L^{\mu a}(x)}{\det L(x)} \quad \frac{L^{\nu b}(\theta)}{\det L(\theta)})\ |0)$$

$$= \sum_{n=0}^{\infty} [\ \eta^{\mu a}\ \eta^{\nu b}\ \mathcal{D}^{o}(n) + \frac{1}{2}(\eta^{\mu\nu}\ \eta^{ab} + \eta^{\mu b}\ \eta^{\nu a}$$

$$- \ \eta^{\mu a}\ \eta^{\nu b})\ \mathcal{D}^{1}(n)\ ]\ (\frac{(-\kappa^2 D(x))^n}{n!}) \qquad (3.14)$$

It is clear that

$$\mathcal{D}^{o}(0) = 1 \qquad \mathcal{D}^{1}(0) = 0 \qquad (3.15)$$

whatever be the parametrization for $L^{\mu a}(x)$. [A closed form expression for $\mathcal{D}^{\mu a,\nu b}$ in the exponential parametrization has been obtained by Ashmore and Delbourgo [13]. In the rational parametrization, in which $L^{\mu a} = \eta^{\mu a} + \frac{1}{2}\kappa\phi^{\mu\ a}$, the expressions for $\mathcal{D}^{o}(n)$ and $\mathcal{D}^{1}(n)$ can be found in Ref.[8]]. Making use of the Sommerfild-Watson transformation,

$$\mathcal{D}^{\mu a,\ \nu b}(x) = \frac{1}{2\pi i}\int_{C_o} dz\ \Gamma(-z)\ [\ \eta^{\mu a}\ \eta^{\nu b}\ \mathcal{D}^{o}(z) + \frac{1}{2}(\eta^{\mu\nu}\ \ \eta^{ab}$$

$$+\ \eta^{\mu b}\ \eta^{\nu a} - \eta^{\mu a}\ \eta^{\nu b})\ \mathcal{D}^{1}(z)]\ (\ \kappa^2 D(x))^{z}, \qquad (3.16)$$

where C_o is a contour enclosing the positive real axis in the clock-wise sense. It can be opened out to be contour C_o' running parallel to the imaginary axis with $-1 < \mathrm{Re}z < 0$.

On the other hand, in our approximation

$$\mathcal{D}^{\mu a,\ \nu b}(x) = \eta^{\mu a}\ \eta^{\nu b}\ \mathcal{D}(x)$$

where $\mathcal{D}(x) \quad = (0\ |T(:e^{-\kappa\chi(x)}:\ :e^{-\ \kappa\chi(\theta)}:)\ |0)$

$$= e^{-\kappa^2 D(x)}$$

$$= \frac{1}{2\pi i}\int_{C_o} dz\ \Gamma(-z)(-\ \kappa^2\ D(x))^{z} \qquad (3.17)$$

This clearly corresponds to replacing $\beta^0(z)$ and $\beta^1(z)$ in (3.16) by their values at z = 0.

We now consider the regularization of a general amplitude through the graviton superpropagator. As the following considerations make it clear, an amplitude which is already finite (without gravity-modification) will remain so when the graviton superpropagator is included. The effect of including the graviton superpropagator would be to give additional terms of O(κ). Also, a single superpropagator in a closed loop is enough to guarantee convergence. In a diagram containing internal loops with more than two vertices, inclusion of a superpropagator between every pair of vertices gives a finite amplitude which differs from the one with a single superpropagator by O(κ). For example, this is the case for the electron-electron-photon vertex correction in the lowest order in e shown in Fig.2. From here onwards, double line with slashes corresponds to the superpropagator.

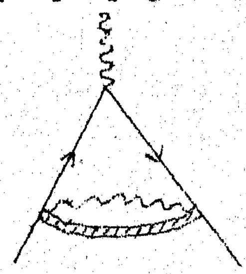

Fig.2 Electron-Electron-Photon Vertex Supergraph.

Consider an amplitude A(p) regularized with a single graviton superpropagator shown in Fig.3.

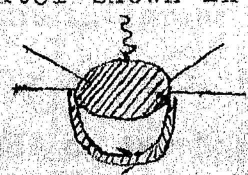

Fig.3 A General Diagram Regularized with one Superpropagator.

This can be written as

$$A(p)= \frac{1}{(2\pi)^4} \int d^4q \; f_{\mu\nu ab}(p,q)\, \tilde{\mathcal{D}}^{\mu a,\nu b}(q)$$

where p symbolically represents all the external momenta, $\sim$ stands for Fourier transformation and $f_{\mu\nu ab}(p,q)$ corresponds to the shaded portion. $f_{\mu\nu ab}(p,q)$ is an integral over internal momenta,

$$f_{\mu\nu ab}(p,q) = \int d^4k_1 \ldots d^4k_r g_{\mu\nu ab}(k_1,\ldots,k_r, p,q) \qquad (3.18)$$

Using the representation (3.16) for $\mathcal{D}^{\mu a, \nu b}$ and Gelfand-Shilov formula (2.8) for Fourier transformation,

$$A(p) = \frac{1}{2\pi i} \int_C dz\, \Gamma(-z)(\kappa^2)^z \, K^{\mu\nu ab}(z)\, f_{\mu\nu ab}(p,z) \qquad (3.19)$$

where $K^{\mu\nu ab}(z)$ stands for the square-bracket containing $\mathcal{D}^0(z)$ and $\mathcal{D}^1(z)$ in eqn. (3.16) and

$$\begin{aligned} f_{\mu\nu ab}(p,z) &= -i\, \frac{\Gamma(2-z)(4\pi)^{2-2z}}{\Gamma(z)} \int \frac{d^4q}{(2\pi)^4}\, f_{\mu\nu ab}(p,q)(-q^2)^{z-2} \\ &= -i\, \frac{\Gamma(2-z)(4\pi)^{2-2z}}{\Gamma(z)} \int \frac{d^4q}{(2\pi)^4}\, d^4k_1 \ldots d^4k_r \\ &\quad g_{\mu\nu ab}(k_1 \ldots k_r, p,q)(-q^2)^{z-2} \end{aligned} \qquad (3.20)$$

To ensure that the integral in (3.20) exists, we have shifted the contour C_0' to C which is parallel to the imaginary axis with $-n-1 < \mathrm{Re}z < -n$ where n is a non-negative integer. This is permissible as the integrand in z, in A(p) does not have any singularities for $\mathrm{Re}z < 0$, before the momentum integrations. The integer n depends on the degree of ultraviolet divergence

in the original amplitude. We have $n = 0$ for logarithmic divergence, $n = 1$ for quadratic divergence and so on. Now we can perform the integrations over the internal momenta.

In the unregularized theory, one would have

$$A(p)_{\text{unreg}} = \eta^{\mu a}\,\eta^{\nu b}\, f_{\mu\nu ab}(p,0) \equiv \eta^{\mu a}\eta^{\mu b}\, f_{\mu\nu ab}(p) \quad (3.21)$$

which is infinite, if the amplitude has ultraviolet divergences. (We ignore infrared divergences to simplify the argument). These ultraviolet infinities are reflected in the singularities of $f_{\mu\nu ab}(p,z)$ in z; in analytic regularization of amplitudes also, one encounters similar singularities [27]. They are in fact simple poles at $z = 0, -1, \ldots, -n$[28].

To evaluate $A(p)$, one folds the contour C towards right so as to enclose the part of the real axis with $\mathrm{Re}\,z \geq -n$ (**this** is permissible [3,4,10]) and evaluate contributions from simple poles at $z = -n, -n+1, \ldots, -1$ and double pole at $z = 0$; (the pole at $z = 0$ is a double pole due to the $\Gamma(-z)$ factor in eqn.(3.19)). Poles at $z \geq 1$ give terms of order $(\kappa^2 \log \kappa)$ and higher. The $\log \kappa$ terms in the renormalization constants in gravity-modified QED come from the double pole at $z = 0$; the equality of these terms in full tensor gravity modified theory and our approximation follows from eqn.(3.15). This explains the fact that Z_2 and Z_3 the electron and photon wavefunction renormalization constants in Refs.[8] and [10] are the same.

When the usual subtractions are made in (3.19) to get the renormalized amplitude, the function $f_{\mu\nu\, ab}(p,z)$ will be

replaced by another function $\bar{f}_{\mu\nu ab}(p,z)$ which has no singularities at $z = 0, -1, \ldots, -n$. The contribution of the simple pole at $z = 0$ will now give (3.21) with $f_{\mu\nu ab}(p,o)$ replaced by $\bar{f}_{\mu\nu ab}(p,o)$. This explains why the renormalized amplitude in our approximation as well as in tensor gravity are the same as in conventional QED [8,10,11].

Till now, we have ignored the terms L_2 and L_3 in eqn. (3.9) or $L_{K.M.}$ in eqn. (3.12) in our approximation which are all formally of $O(\kappa)$. As already mentioned, it would be necessary to include the effects of these terms to regularize certain amplitudes; the photon self-energy supergraph in QED, for example, shown in Fig. 4(a) would now be modified as in Fig. 4(b).

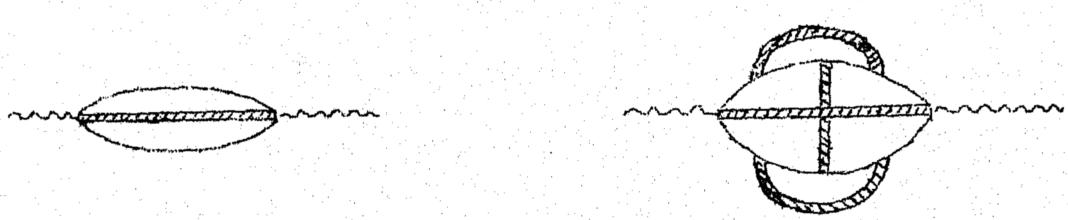

Fig.4 Photon self-energy supergraph.
(a) without including $L_{K.M.}$
(b) with the inclusion of $L_{K.M.}$

However, we find that even in these situations, the renormalized amplitudes to $O(\kappa^0)$, would be unaffected. We expect this to be a general feature, though we do not have a proof to substantiate this.

Isham, et. al.[29] have argued that the damping of ultraviolet infinities due to the non-polynomial nature of the

interactions is strong enough to take care of the self-interactions of gravity also, with a proper choice of the gauge. A detailed analysis of this is lacking. If the argument is proved correct, then with the inclusion of L_{grav} also, a general amplitude would remain finite.

3.2 Regularization of Unrenormalizable Theories

The regularization through the inclusion of gravity is expected to work for nonrenormalizable interactions also. Roughly speaking, the superpropagator $\exp(\frac{\kappa^2}{4\pi^2 x^2})$ as well as $(1/x^2)^n \exp(\frac{\kappa^2}{4\pi^2 x^2}) \to 0$, when $x^2 \to 0$ from the appropriate direction, so that the ultraviolet infinities which are associated with the singularities at $x^2 = 0$ are absent [5].

As an example, consider the self-interaction $\lambda\rho^N$ ($N \geq 3$) of a massless scalar field, ρ , which is unrenormalizable for $N > 4$. The Lagrangian including gravity is:

$$L = \sqrt{(-g)}[\tfrac{1}{2} g^{\mu\nu} \partial_\mu \rho \partial_\nu \rho - \lambda\rho^N] + L_{grav} \tag{3.22}$$

In the approximation of the previous section, the interaction Lagrangian is

$$L_{int} = -\lambda \sqrt{(-g)}\, \rho^N = - \lambda\rho^N e^{-\kappa\chi} \tag{3.23}$$

We shall consider the supergraph for scalar self-energy in the lowest order in λ, (Fig.5) which gives

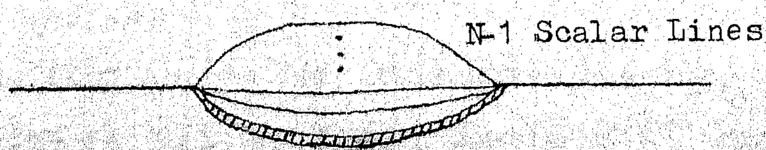

Fig.5 Scalar Self-energy Supergraph.

$$\Pi(k^2) = (-i)N \; N! \, \lambda^2 \int d^4x \; e^{ik.x} \, [D(x)]^{N-1} \, e^{-\kappa^2 D(x)} \tag{3.24}$$

Now

$$[D(x)]^{N-1} \, e^{-\kappa^2 D(x)} = \sum_{n=0}^{\infty} \frac{(-\kappa^2)^n}{n!} [D(x)]^{n+N-1}$$

$$= \frac{1}{2\pi i} \int_{C_0'} dz \, \Gamma(-z) (\kappa^2)^z [D(x)]^{z+N-1} \tag{3.25}$$

To be able to apply the Gelfand-Shilov formula (2.8), we shift the contour in (3.25) to the left so that $0 < \mathrm{Re}(z+N-1) < 2$ and obtain

$$\Pi(k^2) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} dz \, \Gamma(-z) \, \Pi(k^2, z) \tag{3.26}$$

where $-N+2 < \alpha < -N+3$ and

$$\Pi(k^2, z) = \frac{-N \; N!}{16\pi^2} (\kappa^2)^z \frac{\Gamma(3-N-z)}{\Gamma(N+z-1)} \left(\frac{-k^2}{16\pi^2}\right)^{z+N-3} \tag{3.27}$$

On folding the contour in (3.26) to the right on the real axis, we obtain contributions from simple poles at $z = -N+3, -N+4, \ldots -1$ and double poles at $z = 0, 1$ etc. The simple poles give terms of order $(\kappa^2)^{-N+3}, \ldots, \kappa^{-2}$ and the double pole at $z = 0$ gives terms of $O(\log \kappa)$ and $O(\kappa^0)$. The negative powers of κ and $\log\kappa$ terms are reminiscent of the ultraviolet divergences in the self-energy without gravity. κ^{-1} has appeared as an ultraviolet cutoff.

It should be noted that the regularization works even for $N > 4$ corresponding to a non-renormalizable interaction; no new ambiguities appear for this case. However, nonrenormalizability remains in that the higher powers of κ^{-1} cannot be absorbed in the redefinition of a finite number of parameters.

CHAPTER IV

REGULARIZATION OF THEORIES WITH SPONTANEOUS SYMMETRY BREAKING

In this chapter, we consider the regularization of theories with spontaneous symmetry breaking through the inclusion of gravity, taking σ-model as an example. We show explicitly that the counter terms (finite in our case) are the same as those in the symmetric theory. Goldstone theorem and PCAC are verified upto $O(\kappa^0)$.

The relevant known facts about the σ-model are summarised in the first section. Next, we consider the gravity-modified σ-model in the approximation discussed in the previous chapter. In the second section, we present the calculation of vertex correction in the lowest order. The third section deals with the regularization of 'bubble diagrams' in which an internal line closes in on itself. We use this, in Section 4.4 to calculate the self-energies of the mesons and show that the counter-terms are the same as those in the symmetric theory to $O(\kappa^0)$. In the last section, the correction to the vacuum expectation value of σ is computed and it is shown that the Goldstone theorem and PCAC are valid in the lowest order.

4.1 Introduction

The σ-model [30] serves as the field-theoretic realization of current algebra and partial conservation of

axial vector current (PCAC) [31]. Here we consider the $SU(2) \times SU(2)$ σ-model including only the Pion triplet $\vec{\pi}$ and the scalar meson σ. In what follows, we briefly discuss the procedure for renormalization of σ-model as formulated by B.W. Lee et al. [32-34].

The Lagrangian is given by

$$L = L_{sym} + L_b \tag{4.1}$$

where

$$L_{sym} = \frac{1}{2}\left[(\partial_\mu \sigma)^2 + (\partial_\mu \vec{\pi})^2\right] - \frac{1}{2} m^2(\sigma^2 + \vec{\pi}^2) - \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2 \tag{4.2}$$

and

$$L_b(\sigma) = C\,\sigma \tag{4.3}$$

L_{sym} is invariant under $SU(2) \times SU(2)$ transformations, the infinetismal version of which are

$$\begin{aligned} \sigma &\to \sigma - \vec{\beta}\cdot\vec{\pi} \\ \vec{\pi} &\to \vec{\pi} - \vec{\alpha} \times \vec{\pi} + \vec{\beta}\,\sigma \end{aligned} \tag{4.4}$$

where $\vec{\alpha}$ and $\vec{\beta}$ are space-time independent parameters. It is to be noted that the Lagrangian is not normal-ordered. L_b, linear in σ breaks the symmetry and allows a non-vanishing vacuum expectation value of σ . Even when $C = 0$, if $m^2 < 0$, the minimum for the potential would corresponds to a non-zero value of $\langle \sigma \rangle_0$ [32],

$$\langle \sigma \rangle_0 = V \tag{4.5}$$

We may define a new-field σ' by the equation,

$$\sigma = \sigma' + V$$

so that $\langle\sigma'\rangle_0 = 0$ (4.6)

In terms of the translated field (ignoring an inessential constant term),

$$L = \frac{1}{2}[(\partial_\mu\vec{\pi})^2 - m_\pi^2\vec{\pi}^2] + \frac{1}{2}[(\partial_\mu\sigma')^2 - m_\sigma^2\sigma'^2]$$
$$-\lambda V\sigma'(\sigma'^2 + \vec{\pi}^2) - \frac{\lambda}{4}(\sigma'^2 + \vec{\pi}^2)^2 + (C - V m_\pi^2)\sigma' \qquad (4.7)$$

where

$$m_\pi^2 = m^2 + \lambda V^2$$
$$m_\sigma^2 = m^2 + 3\lambda V^2 . \qquad (4.8)$$

With this, the axial current

$$\vec{A}_\mu = \vec{\pi}\partial_\mu\sigma - \sigma\partial_\mu\vec{\pi} \qquad (4.9)$$

would satisfy the PCAC relation,

$$\partial_\mu\vec{A}^\mu = C\vec{\pi} \qquad (4.10)$$

In the tree approximation, the vacuum expectation value of σ, V is given by the solution of the equation,

$$V m_\pi^2 = C$$

or $$V(m^2 + \lambda V^2) = C . \qquad (4.11)$$

When $C = 0$ and $V \neq 0$, (4.11) expresses the Goldstone theorem [35]: the triplet of Pions are the Goldstone bosons. In higher orders, V is given by the solution of the equation,

$$C - V m_\pi^2 + S(V) = 0 \qquad (4.12)$$

where S(V) is the total contribution from the higher order diagrams known as the tadpole diagrams shown in Fig.6.

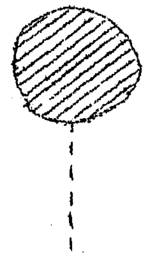

Fig.6 Tadpole Diagram. The dotted line corresponds to the σ'-line. The blob represents the sum of all possible contributions except the tree diagram.

For example, in the one-loop approximation, S(V) is given by the sum of the following two diagrams.

Fig.7 Tadpole Diagrams in One-Loop Order. The solid line corresponds to the π-line.

In the calculation of any physical amplitude, the tadpole diagrams are not considered, if in the expression for the amplitude, the value of V given by the solution of equation (4.12) in the given order is used.

It has been shown that it is possible to eliminate divergences in the spontaneously broken theory described by the Lagrangian in eqn.(4.7), if we choose the counter terms and renormalization constants to be those which render the symmetric theory L_{sym} in eqn.(4.2) finite.

One should also show that the eigenvalue equation for V (eqn.(4.12)) contains only finite quantities after renormalization. The precise form of this is obtained by the identity:

$$\partial^{\mu}(0\,|\,T(A^{i}_{\mu}(x)\;\pi^{j}(0))\,|\,0) = i\,\delta^{ij}(0\,|\sigma|\,0)\,\delta^{4}(x) + C(0\,|\,T(\pi^{i}(x)\;\pi^{j}(0)|0) \tag{4.13}$$

which gives, upon integrating over all space-time:

$$C = V\,[-i\;\Delta^{\pi}_{}(0)]^{-1} \tag{4.14}$$

where $\Delta^{\pi}_{}(k^2)$ is the unrenormalized full Pion-propagator,

$$\delta ij\;\Delta^{\pi}_{}(k^2) = \int e^{ik.x}(0|T(\pi^{i}(x)\pi^{j}(0))|\,0) \tag{4.15}$$

Eqn.(4.14) is the unrenormalized Goldstone theorem: when C=0 and $V \neq 0$, the mass of the Pion vanishes. If we carry out the renormalization on eqn.(4.14),

$$V = Z_M^{\frac{1}{2}}\,V_r$$
$$\Delta^{\pi}_{} = Z_M\,\Delta^{\pi}_{r} \tag{4.16}$$

where Z_M is the wave function renormalization constant in the symmetric theory, then eqn.(4.14) becomes

$$V_r[-\Delta^{\pi}_{r}(0)]^{-1} = \gamma \tag{4.17}$$

where $\gamma = C\;Z_M^{\frac{1}{2}}$

Written in this form, eqn.(4.17) is the renormalized Goldstone theorem. If $\gamma = 0$, either $V = 0$ or the physical Pion mass is zero. The renormalized PCAC equation assumes the form

$$\partial^{\mu}\vec{A}_{\mu}(x) = \gamma\vec{\pi}_{r}(x) \tag{4.18}$$

Gravity-modified σ-model

When couplings with gravity are incorporated, the σ-model Lagrangian would be

$$L' = \sqrt{(-g)}\,[\tfrac{1}{2}g^{\mu\nu}(\partial_\mu\sigma\partial_\nu\sigma + \partial_\mu\vec{\pi}\;\partial_\nu\vec{\pi}) - \tfrac{1}{2}m^2(\sigma^2+\vec{\pi}^2)$$

$$- \frac{\lambda}{4}(\sigma^2+\vec{\pi}^2)^2 + C\sigma] + L_{grav} \qquad (4.19)$$

In our approximation, $g^{\mu\nu}$ is replaced by $\eta^{\mu\nu}$. Then, in terms of the shifted field σ', the Lagrangian would be given by:

$$L'' = L_{free} + L_1' + L_2' + L_3' \qquad (4.20)$$

where

i) L_{free} is the quadratic part of L' given by

$$L_{free} = \frac{1}{2}[(\partial_\mu\vec{\pi})^2 - m_\pi^2\vec{\pi}^2] + \frac{1}{2}[(\partial_\mu\sigma')^2 - m_\sigma^2\sigma'^2] \qquad (4.21)$$

with m_π^2 and m_σ^2 given by eqn.(4.8).

ii) $$L_1' = e^{-\kappa\chi}[-\lambda V\sigma'(\sigma'^2+\vec{\pi}^2) - \frac{\lambda}{4}(\sigma'^2+\vec{\pi}^2)^2] \qquad (4.22)$$

iii) L_2' is the term linear in σ'

$$L_2' = (C - V\,m_\pi^2)\,\sigma' . \qquad (4.23)$$

iv) L_3' is the sum of kinetic-energy modification term and the term linear in σ' modified by gravity:

$$L_3' = \frac{1}{2}(e^{-\kappa\chi}-1)[(\partial_\mu\sigma')^2 + (\partial_\mu\vec{\pi})^2] + (e^{-\kappa\chi}-1)(C - V\,m_\pi^2)\sigma' \qquad (4.24)$$

Again, L_2' is not considered if we are usingthe value of V calculated to the given order.

4.2 Vertex Correction

We consider the corrections to σ'^4 vertex in the lowest order. Other vertices can be treated in the same fashion. Inclusion of L'_3 would not contribute to $O(\kappa^0)$ and we do not consider that here. The relevant Feynman diagrams are shown in Fig.8.

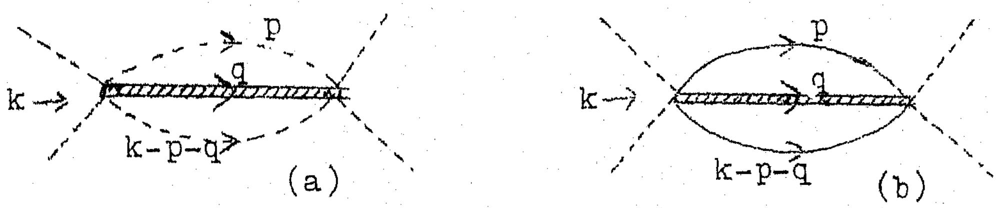

Fig.8 σ'^4 Vertex Correction. Double line with slashes stands for the superpropagator.

Let k be the sum of the incoming momenta. Then the vertex correction is given by

$$\Gamma_{\sigma'^4}(k) = \Gamma^{(a)}_{\sigma'^4}(k) + \Gamma^{(b)}_{\sigma'^4}(k) \tag{4.25}$$

where

$$\Gamma^{(a)}_{\sigma'^4}(k) = \frac{54\,\lambda^2}{(2\pi)^8} \int d^4p\,d^4q \, \frac{1}{(p^2-m_\sigma^2)} \, \frac{1}{\{(k-p-q)^2-m_\sigma^2\}} \, \widetilde{\mathcal{D}}(q) \tag{4.26}$$

and

$$\Gamma^{(b)}_{\sigma'^4}(k) = \frac{18\,\lambda^2}{(2\pi)^8} \int d^4p\,d^4q \, \frac{1}{(p^2-m_\pi^2)} \, \frac{1}{\{(k-p-q)^2-m_\pi^2\}} \, \widetilde{\mathcal{D}}(q) \tag{4.27}$$

with $\widetilde{\mathcal{D}}(q) = \int e^{iq.x} \mathcal{D}(x) = \int e^{iq.x}\, e^{-\kappa^2 D(x)}$

$$= \frac{1}{2\pi i} \int_{C'_0} dz\,\Gamma(-z)(\kappa^2)^z \frac{\Gamma(2-z)}{16\pi^2 i\,\Gamma(z)} \left(\frac{-q^2}{16\pi^2}\right)^{z-2} \tag{4.28}$$

Here C_0, a contour enclosing the positive real axis in the clock-wise sense is opened out to be contour C'_0 running

parallel to the imaginary axis with $-1 < \mathrm{Re}\, z < 0$.

Hence,

$$\Gamma_{\sigma',4}(k) = \frac{18\lambda^2}{(2\pi)^8}\,[3I(k,m_\sigma) + I(k,m_\pi)] \tag{4.29}$$

where

$$I(k,m) = \int d^4p\,d^4q\,\frac{1}{(p^2-m^2)}\,\frac{1}{\{(k-p-q)^2-m^2\}}\quad (q)$$

$$= \frac{1}{2\pi i}\int_{C_0'} dz\;\Gamma(-z)(\kappa^2)^z\,\frac{\Gamma(2-z)}{16\pi^2 i\,\Gamma(z)}\left(-\frac{1}{16\pi^2}\right)^{z-2} I(k,m;z) \tag{4.30}$$

with

$$I(k,m;z) = \int d^4p\,d^4q\,\frac{1}{(p^2-m^2)}\,\frac{1}{\{(k-p-q)^2-m^2\}}(q^2)^{z-2} \tag{4.31}$$

The calculation of $I(k,m;z)$ proceeds as follows [36]:

Using the α-parametrization,

$$\frac{1}{p^2-m^2} = -\int_0^\infty d\alpha\, e^{\alpha(p^2-m^2)}$$

$$\frac{1}{(q^2)^{2-z}} = \frac{(-1)^{2-z}}{\Gamma(2-z)}\int_0^\infty d\alpha\, e^{\alpha q^2}\,\alpha^{1-z}\quad,$$

$$I(k,m;z) = \frac{(-1)^{2-z}}{\Gamma(2-z)}\int_0^\infty d\alpha_1\, d\alpha_2\, d\alpha_3\,\alpha_3^{1-z}\int d^4p\,d^4q$$
$$\exp[\,\alpha_1(p^2-m^2) + \alpha_2\{(k-p-q)^2-m^2\} + \alpha_3 q^2\,]$$

Integrating over the momenta using the relation [1],

$$\int d^4p\; e^{(ap^2+b.p)} = \frac{i\pi^2}{a}\, e^{-b^2/4a}\quad,$$

$$I(k,m;z) = \frac{(-1)^{2-z}}{\Gamma(2-z)}(i\pi^2)^2 \int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3 \frac{\alpha_3^{1-z}}{(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)^2}$$

$$\exp\left[\frac{\alpha_1\alpha_2\alpha_3 k^2}{(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)} - \alpha_1 m^2 - \alpha_2 m^2\right] .$$

Going over to the variables,

$$\alpha_1 = (r\sin\theta\cos\phi)^{-2}, \quad \alpha_2 = (r\sin\theta\sin\phi)^{-2}, \ \alpha_3 = (r\cos\theta)^{-2} ,$$

$$I(k,m;z) = \frac{8(-1)^{2-z}}{\Gamma(2-z)}(i\pi^2)^2 \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \int_0^\infty dr\, r^{2z-1}\sin\phi\cos\phi$$

$$\sin^3\theta\,(\cos\theta)^{2z-1} \exp\left[\frac{1}{r^2}\left(k^2 - \frac{m^2}{\sin^2\theta\sin^2\phi\cos^2\phi}\right)\right]$$

Defining $u = \frac{1}{r^2}$ and integrating over u,

$$I(k,m;z) = \frac{4(-1)^{2-z}}{\Gamma(2-z)}(i\pi^2)^2\,\Gamma(-z) \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \left[\frac{m^2}{\sin^2\theta\sin^2\phi\cos^2\phi} - k^2\right]$$

$$\sin\phi\cos\phi\sin^3\theta\,(\cos\theta)^{2z-1} .$$

Using $x = \sin^2\theta$ as the variable and integrating over x, using the relation [37]

$$\int_0^1 dx(1-x)^{\mu-1}\, x^{\nu-1}(x+\alpha)^{\lambda} = \frac{\alpha^\lambda\,\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)}\ {}_2F_1(-\lambda, \nu; \mu+\nu; -\frac{1}{\alpha}),$$

we get

$$I(k,m;z) = 2(i\pi^2)^2\,\Gamma(-z)\,\Gamma(z) \int_0^{\pi/2} d\phi \sin\phi\cos\phi \left(\frac{-m^2}{\sin^2\phi\cos^2\phi}\right)^z$$

$${}_2F_1(-z, 2-z; 2; \sin^2\phi\cos^2\phi\frac{k^2}{m^2}).$$

Putting $t = \sin^2 2\phi$ and integrating over t using the standard integral [37]:

$$\int_0^1 dt(1-t)^{\mu-1}\, t^{\nu-1}\, {}_2F_1(a_1,a_2;\, b;\, \alpha t)$$

$$= \frac{\Gamma(\mu)\ \Gamma(\nu)}{\Gamma(\mu+\nu)}\ {}_3F_2(\nu, a_1, a_2;\ \mu+\nu,\ b;\ \alpha)$$

we have finally,

$$I(k,m;z) = \frac{-\pi^4}{2}(-4m^2)^z\ \frac{\Gamma(z)\Gamma(-z)\Gamma(\frac{1}{2})\Gamma(1-z)}{\Gamma(\frac{3}{2}-z)}$$

$$ {}_3F_2(1-z,-z,2-z;\ \tfrac{3}{2}-z, 2;\ \frac{k^2}{4m^2}) \tag{4.32}$$

We now substitute (4.32) into (4.30). The contour C_o' is folded back to C_o. The integrand in z has a double pole at z = 0. Evaluating the residue,

$$I(k,m) = -8\pi^6 i[\ 2\log(\frac{\kappa^2 m^2}{4\pi^2}) + f(k,m)] + O(\kappa^2 \log\kappa) \tag{4.33}$$

where

$$f(k,m) = \frac{d}{dz}\{\frac{\Gamma(\frac{1}{2})\ \Gamma(2-z)\ (\Gamma(1-z))^3}{\Gamma(\frac{3}{2}-z)}\ {}_3F_2(1-z,-z,2-z;\tfrac{3}{2}-z,2;\ \frac{k^2}{4m^2})\}\Big|_{z=0} \tag{4.34}$$

It can be easily shown that

$$I(k,m_{\sigma,\pi}) - I(k,m) = O(\lambda V^2) \tag{4.35}$$

Then from eqn.(4.29),

$$\Gamma_{\sigma'^4}(k) = \frac{72\lambda^2}{(2\pi)^8}\ I(k,m) + O(\lambda^3)$$

$$= \frac{-9}{2}\ \frac{i\lambda^2}{\pi^2}\,[\log(\frac{\kappa^2 m^2}{4\pi^2}) + \tfrac{1}{2}f(k,m)\,] + O(\lambda^3) \tag{4.36}$$

Now $\hat{Z}_1$, the vertex renormalization constant is defined by:

$$-6i\lambda + \Gamma_{\sigma'^4}(0) = \hat{Z}_1^{-1}\ (-6i\lambda) \tag{4.37}$$

(We do all renormalizations at k=0, where k stands collectively for the external momenta.)

Hence to $O(\lambda^2)$,

$$\hat{Z}_1 = 1 + \frac{3\lambda^2}{2\pi^2}\left[\log\left(\frac{2\pi}{\kappa m}\right) + O(\kappa^0)\right] \tag{4.38}$$

This coincides with the value of $\hat{Z}_1$ with a cutoff Λ, when Λ is replaced with κ^{-1} [32]*.*

The correction to the charge in the one-loop order, $\delta\lambda$ is given by

$$\delta\lambda = \frac{\Gamma^{(0)}_{\sigma'4}}{-6i} \simeq \frac{12\lambda^2 i}{(2\pi)^8} I(0,m)$$

$$\simeq \frac{3\lambda^2}{2\pi^2}\left[\log\left(\frac{\kappa m}{2\pi}\right) + \frac{1}{4} f(0,m)\right] \tag{4.39}$$

4.3 Regularization of the Bubble Diagram

In the σ-model without gravity-modification, we encounter 'bubble-diagrams' like one shown in Fig.9 as the Lagrangian is not normally-ordered.

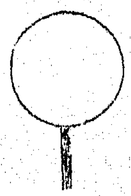

Fig.9 Bubble Diagram.

Here the thick line stands for either two external meson lines or a single σ'-line and the thin line stands for a π-propagator or a σ'-propagator. The contribution of this

* There is a slight error in eqn.(25) of Ref.[32], which should read

$$B_0 = i\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2)^2} \simeq -\frac{1}{8\pi^2}\log\left(\frac{\Lambda}{\mu}\right)$$

would be proportional to

$$\int \frac{d^4p}{p^2 - m^2} \,.$$

To regularize these diagrams, we include the effect of the kinetic-energy modification terms in L_3' and make use of the 'kinking and cradling' procedure devised by Isham, et al.[9]. The bubble diagram would now be modified as shown in Fig. 10.

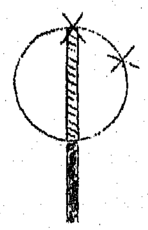

Fig.10 Bubble Diagram in Gravity Modified Theory.

As usual, double line with slashes corresponds to the superpropagator. This diagram represents the sum of diagrams with and without the modifications due to L_3'.

The amplitude would now be proportional to

$$T(m^2) = \frac{1}{(2\pi)^4} \int d^4p\,d^4q\, \frac{1}{(p^2-m^2)} \frac{\{(q+p).p-m^2\}}{\{(q+p)^2-m^2\}} \tilde{\mathcal{D}}(q) \qquad (4.40)$$

Using eqn.(4.28),

$$T(m^2) = \frac{1}{(2\pi)^4} \frac{1}{2\pi i} \int_C dz\, \frac{\Gamma(-z)\Gamma(2-z)(\kappa^2)^z}{16\pi^2 i\, \Gamma(z)} \left(-\frac{1}{16\pi^2}\right)^{z-2} T(m,z) \qquad (4.41)$$

where

$$T(m^2,z) = \int d^4p\,d^4q\, \frac{\{(q+p).p-m^2\}}{(p^2-m^2)\{(q+p)^2-m^2\}} (q^2)^{z-2} \qquad (4.42)$$

In evaluating $T(m^2)$, we have shifted the contour C_0' to C which is parallel to the imaginary axis with $-2 < \mathrm{Re}z < -1$.

No fresh singularities are encontered in shifting the contour. The procedure for the evaluation of $T(m,z)$ is similar to that of $I(k,m,z)$ in Section 4.2. We find,

$$T(m^2,z) = (-1)^z \pi^4 \frac{(\Gamma(1-z))^2 \Gamma(z)\ \Gamma(z-1)}{\Gamma(1-2z)} (m^2)^{z+1} \qquad (4.43)$$

Substituting these in eqn.(4.41) and evaluating the residues at $z = -1$ and $z = 0$, we get:

$$T(m^2) = -i\pi^2 \Big[\Big(\frac{\kappa^2}{16\ \pi^2}\Big)^{-1} + m^2 \log\Big(\frac{\kappa^2 m^2}{16\pi^2}\Big)$$

$$+ m^2 \frac{d}{dz}\Big\{ \frac{(\Gamma(1-z))^4 \Gamma(2-z)}{\Gamma(1-2z)(z+1)} \Big\}\Big|_{z=0} \Big] + O(\kappa^2 \log \kappa) \qquad (4.44)$$

It is interesting to compare this with $\int^{\Lambda} \frac{d^4p}{p^2-m^2}$, where Λ at the top indicates that the integral is cutoff at $|p| = \Lambda$. We find for this, the value

$$-i\ \pi^2 [\ \Lambda^2 - m^2 \log(\ \frac{\Lambda^2}{m^2})\ ].$$

Comparing this with eqn.(4.44) we see that $(\frac{\kappa}{4\ \pi})^{-1}$ is acting as a cutoff in the gravity-modified amplitude.

In the self-energies of π and σ mesons, we encounter $T(m_\pi^2)$ and $T(m_\sigma^2)$. It is convenient to expand these in terms of powers of $(m_\pi^2 - m^2)$ and $(m_\sigma^2 - m^2)$ respectively in discussing various symmetry relations. (m^2 is the mass of the mesons in the symmetric theory.). Consider $T(m_\pi^2)$:

$$T(m_\pi^2) = T(m^2) + (m_\pi^2 - m^2) \frac{\partial}{\partial m^2} T(m^2) + \ldots.. \qquad (4.45)$$

From eqn.(4.8),

$$T(m_\pi^2) = T(m^2) + \lambda V^2 \, T'(m^2) + O(\lambda^2) \qquad (4.46)$$

where

$$T'(m^2) = \frac{\partial}{\partial m^2} T(m^2) = \int \frac{d^4p d^4q}{(2\pi)^4} \left[\frac{1}{(p^2-m^2)^2} \frac{\{(q+p).p-m^2\}}{\{(q+p)^2-m^2\}} + \frac{\{(q+p).p-m^2\}}{(p^2-m^2)\{(q+p)^2-m^2\}^2} - \frac{1}{(p^2-m^2)\{(q+p)^2-m^2\}} \right] \tilde{\mathcal{D}}(q) \qquad (4.47)$$

Using eqn.(4.30),

$$T'(m^2) - \frac{I(0,m)}{(2\pi)^4} = \int \frac{d^4p d^4q}{(2\pi)^4} \frac{1}{(p^2-m^2)\{(q+p)^2-m^2\}} \left[\frac{(q+p).p-m^2}{(p^2-m^2)} + \frac{\{(q+p).p-m^2\}}{\{(q+p)^2-m^2\}} - 2 \right] \tilde{\mathcal{D}}(q) \qquad (4.48)$$

From (4.28), we have

$$T'(m^2) - \frac{I(0,m)}{(2\pi)^4} = \frac{1}{2\pi i} \int_{C_0'} dz \, \Gamma(-z)(\kappa^2)^z \frac{\Gamma(2-z)}{16\pi^2 i \, \Gamma(z)} \int \frac{d^4p d^4q}{(2\pi)^4} \frac{1}{(p^2-m^2)} \frac{1}{\{(q+p)^2-m^2\}} [\ \ldots\ ] \left(\frac{-q^2}{16\pi^2}\right)^{z-2} \qquad (4.49)$$

The momentum integrations are convergent at $z = 0$. Using [16]

$$\lim_{z\to 0} \frac{(-q^2)^{z-2}}{\Gamma(z)} = i \, \pi^2 \, \delta^4(q) \qquad (4.50)$$

and folding back the contour C_0' to C_0, we find that the residue at $z = 0$ vanishes, as the expression inside the square bracket vanishes when $q_\mu = 0$. Hence

$$T'(m^2) = \frac{I(0,m)}{(2\pi)^4} + O(\kappa^2 \log \kappa) \qquad (4.51)$$

Therefore

$$T(m_\pi^2) \approx T(m^2) + \lambda V^2 \frac{I(0,m)}{(2\pi)^4} \qquad (4.52)$$

By identical considerations,

$$T(m_\sigma^2) \simeq T(m^2) + 3\lambda V^2 \frac{I(0,m)}{(2\pi)^4} \qquad (4.53)$$

4.4 <u>Self-Energies of the Mesons</u>

<u>σ-Self-energy</u>

Now, we are equipped to calculate the self-energies of σ and π . The Feynman diagrams contributing to the self-energy of σ are shown in Fig.11.

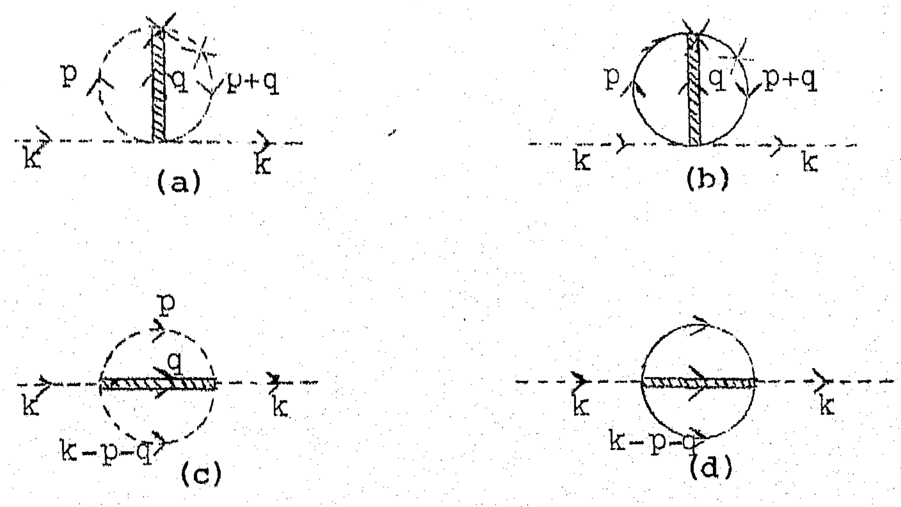

Fig.11 σ-Self-Energy Supergraphs.

$$\Pi_\sigma(k^2) = \Pi_\sigma^{(a)}(k^2) + \Pi_\sigma^{(b)}(k^2) + \Pi_\sigma^{(c)}(k^2) + \Pi_\sigma^{(d)}(k^2) \qquad (4.54)$$

where

$$\Pi_\sigma^{(a)}(k^2) = \frac{3i\lambda}{(2\pi)^4} \int \frac{d^4p\, d^4q}{(2\pi)^4} \frac{\{(p+q).p-m_\sigma^2\}}{(p^2-m_\sigma^2)\{(p+q)^2-m_\sigma^2\}} \mathcal{D}(q)$$

$$= \frac{3i\lambda}{(2\pi)^4} T(m_\sigma^2) \qquad (4.55a)$$

$$\Pi_\sigma^{(b)}(k^2) = \frac{3i\lambda}{(2\pi)^4} \int \frac{d^4p\, d^4q}{(2\pi)^4} \frac{\{(p+q).p-m_\pi^2\}}{(p^2-m_\pi^2)\{(p+q)^2-m_\pi^2\}} \mathcal{D}(q)$$

$$= \frac{3i\lambda}{(2\pi)^4} T(m_\pi^2) \qquad (4.55b)$$

$$\Pi_\sigma^{(c)}(k^2) = \frac{18i\,\lambda^2 V^2}{(2\pi)^8}\int \frac{d^4p\,d^4q}{(p^2-m_\sigma^2)\{(k-p-q)^2-m_\sigma^2\}}\,\widetilde{\mathcal{D}}(q)$$

$$= \frac{18i\;\lambda^2 V^2}{(2\pi)^8}\, I(k,m_\sigma) \qquad (4.55c)$$

$$\Pi_\sigma^{(d)}(k^2) = \frac{6i\,\lambda^2 V^2}{(2\pi)^8}\int \frac{d^4p\,d^4q}{(p^2-m_\pi^2)\{(k-p-q)^2-m_\pi^2\}}\,\widetilde{\mathcal{D}}(q)$$

$$= \frac{6i\lambda^2 V^2}{(2\pi)^8}\, I(k,m_\pi) \qquad (4.55d)$$

Using eqns. (4.52, 4.53), we find for the self-mass δm_σ^2 to $O(\lambda^2)$

$$\delta m_\sigma^2 = \Pi_\sigma(0)$$

$$= \delta m^2 + 3\,\delta\lambda\, V^2 + O(\kappa^2 \log\kappa) \qquad (4.56)$$

where

$$\delta m^2 = \frac{6\,i\,\lambda}{(2\pi)^4}\, T(m^2)$$

$$= \frac{6\lambda}{16\pi^2}\,\Big[\Big(\frac{\kappa^2}{16\pi^2}\Big)^{-1} + m^2 \log\Big(\frac{\kappa^2 m^2}{16\,\pi^2}\Big)$$

$$+ m^2\,\frac{d}{dz}\Big\{\frac{(\Gamma(1-z))^4\,\Gamma(2-z)}{(z+1)\;\Gamma(1-2z)}\Big\}\Big|_{z=0}\Big] + O(\kappa^2\log\kappa) \qquad (4.57)$$

is the self-mass in the in the symmetric theory and $\delta\lambda$ is the correction to charge given by eqn.(4.39).

To this order, the wave-function renormalization constant Z_σ is given by

$$Z_\sigma = 1 + \frac{\partial\Pi_\sigma(k^2)}{\partial k^2}\Big|_{k^2=0} = 1 + \frac{6i\lambda^2 V^2}{(2\pi)^8}\,\frac{\partial}{\partial k^2}\,[\,3I(k,m_\sigma)+I(k,m_\pi)\,]$$

$$= 1 + \frac{3\,\lambda^2 V^2}{16\pi^2}\,\frac{\partial}{\partial k^2}\,[\,3f(k,m_\sigma^2) + f(k,m_\pi^2)]\Big|_{k^2=0} + O(\kappa^2\log\kappa) \qquad (4.58)$$

<u>π-self energy</u>

The Feynman diagrams contributing to the self-energy of π, Π_π, are shown in Fig. 12.

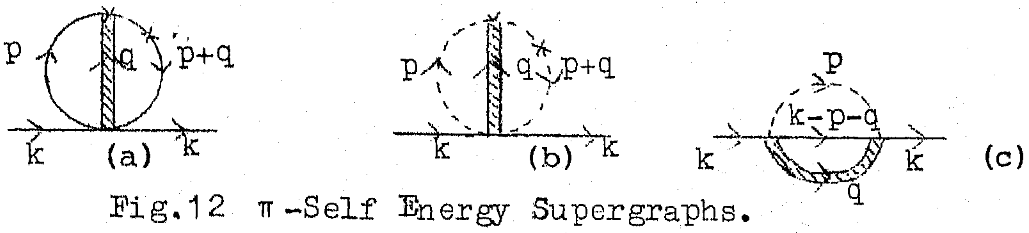

Fig.12 π-Self Energy Supergraphs.

$$\Pi_\pi(k^2) = \Pi_\pi^{(a)}(k^2) + \Pi_\pi^{(b)}(k^2) + \Pi_\pi^{(c)}(k^2) \tag{4.59}$$

where

$$\Pi_\pi^{(a)}(k^2) = \frac{5i\lambda}{(2\pi)^4} \int \frac{d^4p\,d^4q}{(2\pi)^4} \frac{\{(p+q).p-m_\pi^2\}}{(p^2-m_\pi^2)\{(p+q)^2-m_\pi^2\}} \tilde{\mathcal{D}}(q)$$

$$= \frac{5i\lambda}{(2\pi)^4} T(m_\pi^2) \tag{4.60a}$$

$$\Pi_\pi^{(b)}(k^2) = \frac{i\lambda}{(2\pi)^4} \int \frac{d^4p\,d^4q}{(2\pi)^4} \frac{\{(p+q).p-m_\sigma^2\}}{(p^2-m_\sigma^2)\{(p+q)^2-m_\sigma^2\}} \tilde{\mathcal{D}}(q)$$

$$= \frac{i\lambda}{(2\pi)^4} T(m_\sigma^2) \tag{4.60b}$$

and

$$\Pi_\pi^{(c)}(k^2) = \frac{4i\lambda^2 v^2}{(2\pi)^8} \int \frac{d^4p\,d^4q}{(p^2-m_\sigma^2)\{(k-p-q)^2-m_\pi^2\}} \tilde{\mathcal{D}}(q)$$

$$= \frac{4i\lambda^2 v^2}{(2\pi)^8} I(k,m) + O(\lambda^3) \tag{4.60c}$$

Using eqns. (4.52, 4.53), we find for the self-mass δm_π^2 to $O(\lambda^2)$,

$$\delta m_\pi^2 = \delta m^2 + \delta\lambda\, v^2 + O(\kappa^2 \log \kappa)$$

$$= \frac{6i\lambda}{(2\pi)^4} T(m^2) + \frac{12i\lambda^2 v^2}{(2\pi)^8} I(0,m) + O(\kappa^2 \log \kappa) \tag{4.61}$$

To this order, the wave-function renormalization constant, Z is given by

$$Z_\pi = 1 + \left.\frac{\partial \Pi_\pi(k^2)}{\partial k^2}\right|_{k^2=0} = 1 + \frac{4i\,\lambda^2 V^2}{(2\pi)^8}\,\frac{\partial}{\partial k^2}\, I(k,m)$$

$$= 1 + \frac{\lambda^2 V^2}{8\pi^2}\,\frac{\partial}{\partial k^2}\,f(k,m)\Big|_{k^2=0} + O(\kappa^2 \log\kappa). \qquad (4.62)$$

From eqns.(4.56) and (4.61) we find that the self-masses δm_σ^2 and δm_π^2 are consistent with the relations in eqn.(4.8) to $O(\kappa^0)$: the self-masses are the same as the ones got by computing the self mass and the self-charge in the symmetric theory and using eqn.(4.8). Z_σ and Z_π are of $O(\kappa^0)$ corresponding to the fact that in the original σ-model they are finite in this order.

4.5 Goldstone Theory and PCAC

Goldstone theorem

The Feynman diagrams in Fig.12 contribute to $V = \langle\sigma\rangle_0$ in the lowest order

Fig.12 Corrections to $\langle\sigma\rangle_0$ in the Lowest Order.

In the notation of Section 4.1,

$$S(V) = \frac{-3i\lambda}{(2\pi)^4}\,[T(m_\pi^2) + T(m_\sigma^2)] \qquad (4.63$$

Using eqns. (4.52, (4.53) and (4.61), we find to $O(\lambda^2)$,

$$S(V) = \frac{-3i\lambda}{(2\pi)^4}\,[2T(m^2) + 4\lambda V^2 \frac{I(0,m)}{(2\pi)^4}]$$

$$= -\delta m_\pi^2 + O(\kappa^2 \log\kappa). \qquad (4.64)$$

Now from the considerations of Section 4.1, V is a solution of the equation,

$$(C - V\, m_\pi^2\ ) + S(V) = 0 \tag{4.65}$$

Hence $V(m_\pi^2 + \delta m_\pi^2\ ) + O(\ \kappa^2 \log \kappa) = C$

$$\text{i.e.}\quad V[\ -i\ \Delta^\pi(0)^{-1} + O(\ \kappa^2 \log \kappa)] = C \tag{4.66}$$

where $\Delta^\pi(k^2)$ is the unrenormalized full Pion propagator upto $O(\ \lambda^2)$. Comparing this with eqn.(4.14), we find that the Goldstone theorem is validated upto $O(\ \kappa\ )$: when $C = 0$ and $V\neq 0$, the Pion mass vanishes.

PCAC

The Pion-axial vector vertex Γ_μ^5 is defined through:

$$\int (0|T(A_\mu^\alpha(x)\ \pi^\beta(0))|0)\ e^{ik.x}\ d^4x = \Gamma_\mu^5(k)\ \Delta^\pi(k^2)\delta_{\alpha\beta} \tag{4.67}$$

Then from eqn.(4.13) which holds in the gravity-modified theory to $O(\kappa^0)$, we find

$$\begin{aligned} k_\mu\ \Gamma^{5\mu}(k) &= -\ \langle\sigma\rangle_0[\Delta^\pi\ (k^2)\ ]^{-1} + iC + 0(\kappa) \\ &= -V[\ \Delta^\pi(k^2)\ ]^{-1} + iC + 0(\kappa) \end{aligned} \tag{4.68}$$

Using eqn.(4.66) expressing Goldstone theorem,

$$k_\mu\Gamma^{5\mu}(k) = -V\ \ \Delta^\pi(k^2)^{-1} - \Delta^\pi(0)^{-1} + O(\kappa) \tag{4.69}$$

In the tree-graph approximation, $\Gamma^{5\mu} = iVk^\mu$ and the PCAC relation (4.69) is trivially satisfied. We now show that at the 'one-loop' level also this holds. In this order, the following Feynman diagram contributes to $\Gamma^{5\mu}(k)$ [Fig.13].

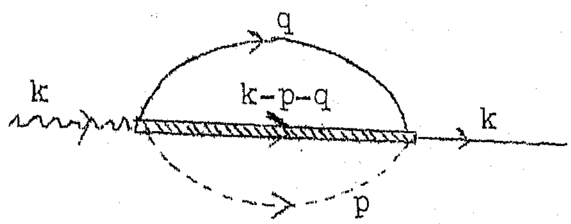

Fig.13 'One Loop' Contribution to $\Gamma^{5\mu}$. Here wavy-line represents the axial vector current.

Then

$$\Gamma^{5\mu}(k) = iVk^{\mu} + \frac{2\lambda V}{(2\pi)^8} \int d^4p d^4q \frac{(p-q)_\mu}{(p^2-m_\sigma^2)(q^2-m_\pi^2)} \tilde{\mathcal{D}}(k-p-q) \tag{4.70}$$

Using $m_\sigma^2 - m_\pi^2 = 2\lambda V^2$,

$$\Gamma^{5\mu}(k) = iVk^{\mu} + \frac{2\lambda V}{(2\pi)^8} \int \frac{d^4p d^4q \ (p-q)_\mu}{(q^2-m_\pi^2)(p^2-m_\pi^2)} \tilde{\mathcal{D}}(k-p-q)$$

$$+ \frac{4\lambda^2 V^3}{(2\pi)^8} \int \frac{d^4p d^4q \ (p-q)_\mu}{(p^2-m_\pi^2)} \frac{1}{(p^2-m_\sigma^2)(q^2-m_\pi^2)} \tilde{\mathcal{D}}(k-p-q)$$

The second term on the R.H.S. vanishes. Changing the variable of integration in the third term,

$$k_\mu \Gamma^{5\mu}(k) = iVk^2 + \frac{4\lambda^2 V^3}{(2\pi)^8} \int d^4p d^4q \frac{(2p+q-k).k}{(p^2-m_\sigma^2)} \frac{1}{(p^2-m_\pi^2)\{(k-p-q)^2-m_\pi^2\}} \tilde{\mathcal{D}}(q)$$

Using arguments similar to those used in Section 4.3,

$$\int \frac{d^4p d^4q}{(p^2-m_\sigma^2)} \left[\frac{(2p+q-k).k}{(p^2-m_\pi^2)\{(k-p-q)^2-m_\pi^2\}} - \frac{(2k.(p+q)-k^2)}{\{(k-p-q)^2-m_\pi^2\}\{(p+q)^2-m_\pi^2\}} \right] \tilde{\mathcal{D}}(q)$$

$$= O(\kappa^2 \log \kappa)$$

Hence

$$k_\mu \Gamma^{5\mu}(k) = iVk^2 + \frac{4\lambda^2 V^3}{(2\pi)^8} \int d^4p d^4q \frac{1}{(p^2-m_\sigma^2)} \frac{\{2k.(p+q)-k^2\}\tilde{\mathcal{D}}(q)}{\{(k-p-q)^2-m_\pi^2\}\{(p+q)^2-m_\pi^2\}}$$

$$+ O(\kappa^2 \log \kappa)$$

$$= iVk^2 + \frac{4\lambda^2 V^3}{(2\pi)^8} \int d^4p\, d^4q\, \frac{1}{(p^2-m_\sigma^2)}\left[\frac{1}{\{(k-p-q)^2-m_\pi^2\}} - \frac{1}{\{(p+q)^2-m_\pi^2\}}\right]\tilde{\beta}(q) + O(\kappa^2 \log\kappa)$$

Using eqn.(4.60),

$$k_\mu \Gamma^{5\mu}(k) = iVk^2 - i[\Pi_\pi(k^2) - \Pi_\pi(0)] + O(\kappa^2 \log \kappa)$$

$$\therefore\ k_\mu \Gamma^{5\mu}(k) = -V[\Delta^\pi(k^2)^{-1} - \Delta^\pi(0)^{-1}] + O(\kappa^2 \log\kappa) \quad (4.71)$$

Comparing this with eqn. (4.69), we find that the PCAC relation is valid in perturbation theory to $O(\kappa^0)$.

CHAPTER V

GRAVITY-MODIFIED NON-ABELIAN GAUGE THEORIES

In this chapter we will consider the regularization of non-Abelian gauge theories by gravity in the approximation discussed in Chapter III. First, we discuss the regularization of pure Yang-Mills fields in the lowest order in the coupling constant. It will be seen that the result is gauge-invariant only upto $O(\log\kappa)$ and not to $O(\kappa^0)$. In Section 5.2, this is remedied by using the 'point-splitting method' for the definition of the current operator. In Section 5.3, we present the calculation of renormalization constants in quantum chromodynamics. In the last section, we briefly discuss the regularization of spontaneously broken gauge theories.

5.1 Pure Yang-Mills Fields

The Yang-Mills group [38] is assumed to be a compact semisimple group G. In the absence of gravity, the Lagrangian in an obvious notation is[39]:

$$L_{Y.M.} = -\frac{1}{4}F^a_{\mu\nu}\, F^{a\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^{\mu a})^2 + \partial^\mu C^{\dagger a}\, \partial_\mu C^a + gf^{abc}\,\partial^\mu C^{\dagger a} A^b_\mu\, C^c \qquad (5.1)$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_{\mu\nu} - \partial_\nu A^a_\mu + gf^{abc} A^b_\mu A^c_\nu.$$

Here C^a and $C^{\dagger a}$ are the familiar ghost-fields and α is a real parameter fixing the gauge. When gravity is included, we have the following Lagrangian:

$$L = \sqrt{(-g)}\,[\, -\frac{1}{4} g^{\mu\lambda}\, g^{\nu\rho}\, F^a_{\mu\nu}\, F^a_{\lambda\rho} - \frac{1}{2\alpha}\, g^{\mu\nu}\, g^{\lambda\rho}\, A^a_{\nu;\mu}\, A^a_{\rho;\lambda}$$

$$+ g^{\mu\nu}\, \partial_\mu C^{\dagger a} \partial_\nu\, C^a + g f^{abc}\, g^{\mu\nu}\, \partial_\mu C^{\dagger a} A^b_\nu\, C^c\,]$$

$$+ L_{grav} + L_{gauge}(\text{gravity}) \qquad (5.2)$$

where $A_{\mu;\nu}$ are the usual general covariant derivatives.

In the afore-mentioned approximation, L is to be replaced by the Lagrangian:

$$L' = L_1 + L_{K.M.} \qquad (5.3)$$

where

$$L_1 = L_{free} + e^{-\kappa\chi}[-g\, f^{abc}\, \partial^\mu A^{a\nu}\, A^b_\mu\, A^c_\nu$$

$$- \frac{1}{4}\, g^2 f^{abc} f^{ab'c'}\, A^b_\mu\, A^c_\nu\, A^{b'\mu} A^{c'\nu} + g f^{abc}\, \partial^\mu C^{\dagger a}\, A^b_\mu\, C^c] \qquad (5.4)$$

and

$$L_{K.M.} = (e^{-\kappa\chi} - 1)[\, -\frac{1}{4}(\, \partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\alpha}\, (\partial^\mu\, A^a_\mu)^2$$

$$+ \partial^\mu C^{\dagger a}\, \partial_\mu C^a] \qquad (5.5)$$

The raising and lowering of indices in (5.4) and (5.5) is understood to be done by the Minkowski metric. Here $e^{-\kappa\chi}$ stands for the normal ordered expression. The ordering of the Yang-Mills part will be discussed at the appropriate stage. For the calculation of renormalization constants in the following subsections, we will ignore $L_{K.M.}$ which is formally of $O(\kappa)$.

Our propagators are:

$$D^{ab}_{\mu\nu}(x) = (0|\, T(A^a_\mu\, (x) A^b_\nu(0))|\, 0) = -\delta^{ab}[\, \eta_{\mu\nu} - (1-\alpha)\frac{\partial_\mu \partial_\nu}{\partial^2}\,]\, D(x)$$

$$(0\, |T(C^a(x)\, C^{\dagger b}(0))\,|\, 0) = \delta^{ab} D(x) \qquad (5.6)$$

and the graviton superpropagator, for which we shall, for the time being, employ the expression in the Fock-deDonder gauge given by eqn. (3.17). The effect of introducing a more general gauge for the graviton propagator will be considered at an appropriate place.

We shall now consider the primitive divergences of the Yang-Mills theory, calculate the renormalization constants to $O(\log \kappa)$ and verify that they are consistent with the Ward-identities [40].

5.1(a) Self-energy of the gauge particles

A direct momentum-space calculation gives a non-gauge invariant result with a longitudinal part in $\Pi^{\mu\nu}$ proportional to κ^{-2}. [This is analogous to the situation regarding the photon self-energy in Refs.[8] and[10]]. Following Ref. [9] we shall present a coordinate space calculation employing the 'calculus derivatives' and the normal-ordered expression for $L_{Y.M.}$. The relevant lowest order supergraphs are shown in Fig. 14.

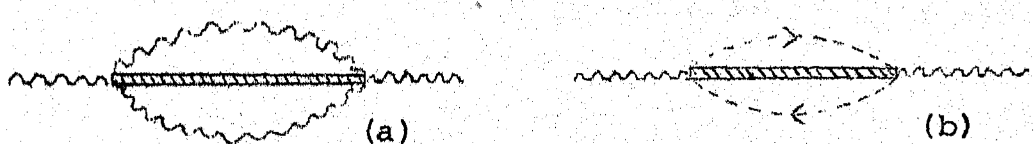

Fig.14 Self-Energy Supergraphs for the Gauge Particles. Wavy-lines denote the gauge particles and dotted line, the ghost-field.

Denoting the contributions from the two graphs by $\Pi^{\mu\nu(a)}$ and $\Pi^{\mu\nu(b)}$, we have, after a straightforward calculation:

$$\Pi^{\mu\nu(a)}(x) = -ig^2 C_2(G)[-\eta^{\mu\nu}\partial^2(D^2\mathcal{D})+\partial^\mu\partial^\nu(D^2\mathcal{D})$$
$$+ 2\partial^\mu(\partial^\nu DD\mathcal{D})-2\eta^{\mu\nu}\partial_\lambda(\partial^\lambda DD\mathcal{D}) -2(\partial^\mu\partial^\nu D)D\mathcal{D}$$
$$+ 3\partial^\mu D\partial^\nu D\mathcal{D} - \eta^{\mu\nu}\partial^2 DD\mathcal{D}$$
$$+ \alpha'\partial^\lambda\partial^\rho\{\eta^{\mu\nu}(\partial_\lambda\partial_\rho D/\partial^2)D\mathcal{D} + \eta_{\lambda\rho}(\partial^\mu\partial^\nu D/\partial^2)D\mathcal{D}$$
$$- \eta^\mu_\rho(\partial^\nu\partial_\lambda D/\partial^2)D\mathcal{D} - \eta^\mu_\lambda(\partial_\rho\partial^\nu D/\partial^2)D\mathcal{D}\}$$
$$- 2\alpha'\partial^\lambda\{\eta^\mu_\lambda\partial^\rho D(\partial^\nu\partial_\rho D/\partial^2)\mathcal{D} - \eta^{\mu\nu}\partial^\rho D(\partial_\lambda\partial_\rho D/\partial^2)\mathcal{D}\}$$
$$+ \alpha'\{\eta^{\lambda\rho}\partial^\mu\partial^\nu D(\partial_\lambda\partial_\rho D/\partial^2)\mathcal{D} - \eta^{\lambda\mu}\partial^\rho\partial^\nu D(\partial_\lambda\partial_\rho D/\partial^2)\mathcal{D}$$
$$- \eta^{\lambda\nu}\partial^\rho\partial^\mu D(\partial_\lambda\partial_\rho D/\partial^2)\mathcal{D} + \eta^{\mu\nu}\partial^\lambda\partial^\rho D(\partial_\lambda\partial_\rho D/\partial^2)\mathcal{D}\}$$
$$- \alpha'^2\partial^\lambda\partial^\rho\{(\partial_\lambda\partial_\rho D/\partial^2)(\partial^\mu\partial^\nu D/\partial^2)\mathcal{D} - (\partial^\mu\partial_\lambda D/\partial^2)$$
$$(\partial^\nu\partial_\rho D/\partial^2)\mathcal{D}\}] \text{, where } \partial_\mu\partial_\nu D/\partial^2 \equiv (\partial_\mu\partial_\nu/\partial^2)D \qquad (5.7a)$$

and

$$\Pi^{\mu\nu(b)}(x) = ig^2 C_2(G)\, \partial^\mu D\, \partial^\nu D\mathcal{D}(x) \qquad (5.7b)$$

Here we have put $\alpha' = 1-\alpha$. $C_2(G)$ is the quadratic Casimir invariant in the adjoint representation, defined by $\sum_{c,d} f^{acd} f^{bcd} = C_2(G)\,\delta^{ab}$.

Writing the Sommerfeld-Watson transform for $\mathcal{D}$ (eqn.3.17) and employing the formulae of the calculus of derivatives, in eqns. (2.19) to (2.21) and eqns.(2.24) to (2.27), we have

$$\Pi^{\mu\nu}(x) = \frac{1}{2\pi i}\int_{C_0} dz\, \Gamma(-z)\, \Pi^{\mu\nu}(x;\,z) \qquad (5.8)$$

where

$$\Pi^{\mu\nu}(x;z) = -ig^2 C_2(G)(\kappa^2)^z [\{1+\frac{8-\alpha'^2 z}{4(z+2)} - \frac{(2-3\alpha')}{(z+2)(z+3)}\}$$
$$\times(\partial^\mu\partial^\nu-\eta^{\mu\nu}\partial^2) + \frac{1}{2}\,\frac{(3\alpha'-2)z\,\eta^{\mu\nu}\partial^2}{(z+1)(z+2)(z+3)}]\, D(x)^{z+2} \qquad (5.9)$$

Its Fourier transform employing (2.8) is :

$$\Pi^{\mu\nu}(k;z) = \int e^{-ik.x}\, \Pi^{\mu\nu}(x,z)\, d^4x$$
$$= \frac{g^2 C_2(G)}{16\,\pi^2} \frac{\Gamma(-z)}{\Gamma(z+2)} \left(- \frac{\kappa^2 k^2}{16\,\pi^2}\right)^z$$
$$[\{1+ \frac{(8-\alpha'^2 z)}{4(z+2)} - \frac{(2-3\,\alpha')}{(z+2)(z+3)}\}\,(k^\mu k^\nu - \eta^{\mu\nu}k^2)$$
$$+ \frac{1}{2}\,\frac{(3\,\alpha'-2)z}{(z+1)(z+2)(z+3)}\,\eta^{\mu\nu}\,k^2\,] \qquad (5.10)$$

Here onwards, it is assumed that the contour C_o is stretched to lie parallel to the imaginary axis between 0 and -1 before performing Fourier transform, integrations, etc. and folded back afterwards, without explicitly mentioning it.

The integrand in

$$\Pi^{\mu\nu}(k) = \frac{1}{2\pi i} \int_{C_o} dz\, \Gamma(-z)\, \Pi^{\mu\nu}(k;z) \qquad (5.11)$$

clearly has double poles at $z = 0,1,2,\ldots.$ Evaluating the residue at $z=0$, we get

$$\Pi^{\mu\nu}(k) = \frac{-g^2 C_2(G)}{16\,\pi^2}\,[\,(k^\mu k^\nu - \eta^{\mu\nu}k^2)(\tfrac{5}{3} + \tfrac{\alpha'}{2})\log(\frac{-\kappa^2 k^2}{16\,\pi^2})$$
$$+ \frac{d}{dz}\{\, \frac{(\Gamma(1-z))^2}{\Gamma(z+2)}\,(1+ \frac{(8-\alpha'^2 z)}{4(z+2)} - \frac{(2-3\alpha')}{(z+2)(z+3)}).$$
$$\times\,(k^\mu k^\nu - \eta^{\mu\nu}k^2) + \frac{1}{2}\,\frac{(3\alpha'-2)z\,\eta^{\mu\nu}\,k^2}{(z+1)(z+2)(z+3)}\}\Big|_{z=o}\,]$$
$$+\,O(\kappa^2 \log \kappa^2) \qquad (5.12)$$

The presence of $\eta^{\mu\nu}$ term shows lack of gauge invariance. However, it should be noted that $\Pi^{\mu\nu}(0)=0$. This implies that the gauge quantum does not acquire mass.

The renormalization constant Z_3 is defined through the relation:

$$D^{tr\,ab}_{un\,\mu\nu}(k)\Big|_{k^2=-\mu^2} = \frac{i}{\mu^2} Z_3(\eta_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2})\,\delta^{ab} \tag{5.13}$$

where $D^{tr\,ab}_{un\,\mu\nu}(k)$ is the transverse part of the unrenormalized propagator for the gauge particles. To $O(g^2)$,

$$D^{ab}_{un\,\mu\nu}(k) = D^{ab}_{\mu\nu}(k) - iD^{ac}_{\mu\lambda}(k)\;\Pi^{\lambda\rho}(k)\;D^{cb}_{\lambda\nu}(k) \tag{5.14}$$

This gives,

$$Z_3 = 1 + \frac{g^2}{16\pi^2} C_2(G)\;[(\frac{13}{3} - \alpha)\log(\frac{4\pi}{\kappa\mu}) + O(\kappa^0)] \tag{5.15}$$

5.1(b) Ghost self-energy

The lowest order supergraph contributing to the ghost self-energy is shown in Fig.15.

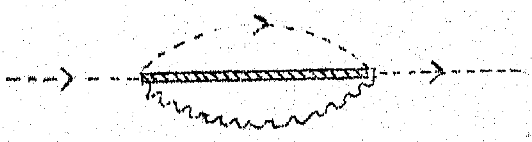

Fig.15 Ghost Self-Energy Supergraph.

We have,

$$\Pi_G(x^2) = -ig^2C_2(G)\partial^\mu[\,(\eta_{\mu\nu} - \alpha'\,\frac{\partial_\mu\partial_\nu}{\partial^2})D\,\partial^\nu D\mathcal{D}] \tag{5.16}$$

Using eqns. (3.17), (2.19) and (2.24)

$$\Pi_G(x^2) = -ig^2C_2(G)\frac{(1+\frac{1}{2}\alpha')}{2\pi i}\int_{C_0} dz\,\frac{\Gamma(-z)}{(z+2)}(\kappa^2)^z\;\partial^2[D^{2+z}] \tag{5.17}$$

Taking the Fourier transform using (2.8),

$$\Pi_G(k^2) = \int e^{-ik.x}\,\Pi_G(x^2)dx$$

$$= \frac{g^2C_2(G)k^2(1+\frac{1}{2}\alpha')}{16\pi^2}\;\frac{1}{2\pi i}\int_{C_0} dz\,\frac{\Gamma(-z)\,\Gamma(-z)}{\Gamma(z+3)}\left(\frac{-\kappa^2k^2}{16\pi^2}\right)^z \tag{5.18}$$

Evaluating the double pole at $z = 0$, we have

$$\Pi_G(k^2) = -\frac{g^2C_2(G)}{32\pi^2}\left(\frac{3}{2}-\frac{1}{2}\alpha\right)k^2\left[\log\left(\frac{-\kappa^2k^2}{16\pi^2}\right) + O(\kappa^0)\right] \tag{5.19}$$

Defining the renormalization constant $\tilde{Z}_3$ by

$$D_{un}^{G^{ab}}(-\mu^2) = \left.\frac{i\,\delta^{ab}}{k^2 - \Pi_G(k^2)}\right|_{k^2=-\mu^2} \equiv \frac{-i}{\mu^2}\tilde{Z}_3\;\delta^{ab} \tag{5.20}$$

We have finally,

$$\tilde{Z}_3 = 1 + \frac{g^2C_2(G)}{16\pi^2}\left[\left(\frac{3}{2}-\frac{1}{2}\alpha\right)\log\left(\frac{4\pi}{\kappa\mu}\right) + O(\kappa^0)\right] \tag{5.21}$$

5.1(c) The Ghost-Ghost-Vector Vertex

The relevant lowest order supergraphs are shown in Fig. 16.

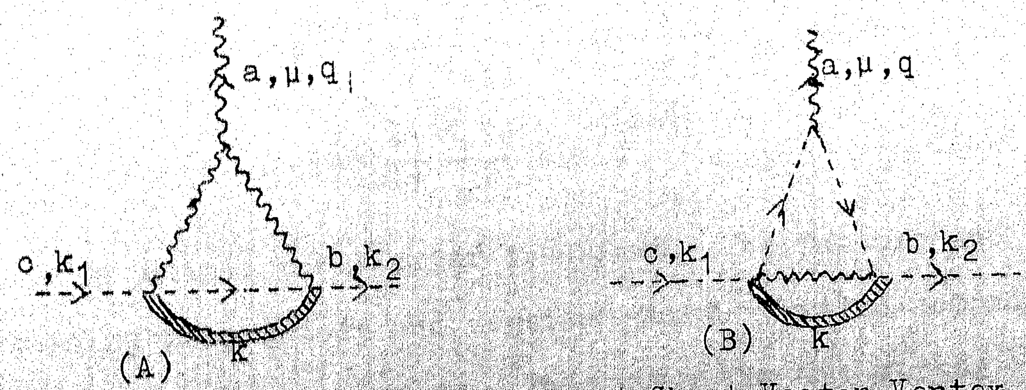

Fig.16 Corrections to Ghost-Ghost-Vector Vertex.

Strictly speaking, there should be a superpropagator between every pair of vertices. It turns out, however, that it is sufficient to include one superpropagator to secure ultra-violet convergence; the contribution of the graphs with three superpropagators will differ from those in Fig. 16 by terms of $O(\kappa^2 \log\kappa)$ and higher.

It is convenient to combine with the superpropagator, the ghost propagator in Fig. 16(A) and the vector propagator in Fig. 16(B) and use the calculus of derivatives as before, i.e., we write

$$\mathcal{F}(D\, e^{-\kappa^2 D})(k) = \frac{1}{2\pi i}\int_{C_0} dz\; \Gamma(-z)(\kappa^2)^z\, \mathcal{F}(D^{z+1})$$

$$= \frac{i}{2\pi i}\int_{C_0} dz\; \frac{\Gamma(-z)\,\Gamma(1-z)}{\Gamma(1+z)}\left(-\frac{\kappa^2}{16\pi^2}\right)^z (k^2)^{z-1} \tag{5.22}$$

and

$$\mathcal{F}\Big[\,(\eta_{\mu\nu} - \alpha' \frac{\partial_\mu\partial_\nu}{\partial^2})D\, e^{-\kappa^2 D}\Big](k) = \frac{1}{2\pi i}\int_{C_0} dz\; \Gamma(-z)(\kappa^2)^z$$

$$\mathcal{F}\Big\{\,(\partial_\mu\partial_\nu - \alpha' \frac{\partial_\mu\partial_\nu}{\partial^2})D\;\; D^z\Big\}\;(k)$$

$$= \frac{i}{2\pi i}\int_{C_0} dz\; \frac{\Gamma(-z)\,\Gamma(1-z)}{\Gamma(1+z)}\left(-\frac{\kappa^2}{16\,\pi^2}\right)^z \Big[\Big(1 - \frac{\alpha'}{2}\,\frac{z}{z+1}\Big)\eta_{\mu\nu} + \alpha'\frac{(z-1)}{(z+1)}\,\frac{k_\mu k_\nu}{k^2}\,\Big](k^2)^{z-1} \tag{5.23}$$

where $\mathcal{F}$ stands for Fourier transform. The factors for the remaining propagators and vertices can be written according to the usual Feynman rules. This gives for the vertex correction,

$$\tilde{\Lambda}^{\mu\, abc}(k_1,k_2,q) = \frac{1}{2\pi i}\int_{C_0} dz\, \Gamma(-z)\, [\tilde{\Lambda}^{\mu\, abc^{(A)}}(k_1,k_2,q;z) + \tilde{\Lambda}^{\mu\, abc^{(B)}}(k_1,k_2,q;z)] \tag{5.24}$$

where

$$\tilde{\Lambda}^{\mu\, abc^{(A)}}(k_1,k_2,q;z) = \frac{i}{2}g^3 C_2(G)\, f^{abc}\, k_2^{\delta}\frac{\Gamma(1-z)}{\Gamma(z+1)}\left(\frac{-\kappa^2}{16\pi^2}\right)^z$$

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2)^{1-z}}\frac{1}{(k-k_2)^2}\frac{1}{(k-k_1)^2}\, k^{\gamma}\, [\eta^{\mu\nu}(-k+q+k_1)^{\lambda}$$

$$+ \eta^{\nu\lambda}(2k-k_1-k_2)^{\mu} + \eta^{\lambda\mu}(-k+k_2-q)^{\nu}]\, [\eta_{\nu\gamma}\,\eta_{\lambda\delta}$$

$$- \alpha'\frac{(k-k_2)_{\lambda}(k-k_2)_{\delta}\eta_{\nu\gamma}}{(k-k_2)^2} - \alpha'\frac{(k-k_1)_{\nu}(k-k_1)_{\gamma}\eta_{\lambda\delta}}{(k-k_1)^2}$$

$$+\alpha'^2\frac{(k-k_1)_{\nu}(k-k_1)_{\gamma}(k-k_2)_{\lambda}(k-k_2)_{\delta}}{(k-k_1)^2(k-k_2)^2}\,] \tag{5.25}$$

and

$$\tilde{\Lambda}^{\mu\, abc^{(B)}}(k_1,k_2,q;z) = -\frac{ig^3}{2}C_2(G)f^{abc}\, k_2^{\delta}\,\frac{\Gamma(1-z)}{\Gamma(1+z)}\left(\frac{-\kappa^2}{16\pi^2}\right)^z$$

$$\int\frac{d^4k}{(2\pi)^4}\frac{(k+k_1)^{\lambda}(k+k_2)^{\mu}}{(k+k_1)^2(k+k_2)^2(k^2)^{1-z}}\,[\,(1-\frac{z\alpha'}{2(z+1)})\,\eta_{\lambda\delta}$$

$$+\frac{\alpha'(z-1)}{(z+1)}\frac{k_{\lambda}k_{\delta}}{k^2}\,] \tag{5.26}$$

Consider $\tilde{\Lambda}^{\mu\, abc^{(A)}}$. In the calculation of the renormalization constant to $O(\log\kappa)$, we need to consider only terms in $\tilde{\Lambda}^{\mu\, abc^{(A)}}(k_1,k_2,q;z)$ which have a pole at $z = 0$(see eqn.(5.24)). Those terms which tend to zero faster than k^{-4} as $k\to\infty$ in the momentum integrand do not contribute to the pole at $z=0$ after momentum integration, and can be ignored. We have then,after simplifications,

$$\tilde{\Lambda}^{\mu abc^{(A)}}(k_1,k_2,q;z) \simeq \frac{i\alpha}{2}\, g^3 C_2(G) f^{abc} k_2^{\delta}\, \frac{\Gamma(1-z)}{\Gamma(z+1)}\left(\frac{-\kappa^2}{16\pi^2}\right)^z$$

$$\int \frac{d^4k}{(2\pi)^4}\; \frac{(k^{\mu} k_{\delta} - \eta^{\mu}_{\delta} k^2)}{(k^2)^{1-z}(k-k_2)^2(k-k_1)^2} \qquad (5.27)$$

We combine the denominators through auxiliary parameters [41],

$$\frac{1}{(k^2)^{1-z}}\,\frac{1}{(k-k_1)^2}\,\frac{1}{(k-k_2)^2} = \frac{\Gamma(3-z)}{\Gamma(1-z)} \int_0^1 dx \int_0^{1-x} dy\,(1-x-y)^{-z}$$

$$[\,(k-k_2)^2 x + (k-k_1)^2 y + k^2(1-x-y)\,]^{z-3} \qquad (5.28)$$

and do momentum integrations, using [42]

$$\int d^4k\, \frac{k^{\mu} k^{\nu}}{(k^2+2k.p+m^2)^{\alpha}} = \frac{i\,\pi^2}{(m^2-p^2)^{\alpha-2}}\;\frac{1}{\Gamma(\alpha)}$$

$$\{\Gamma(\alpha-2)\; p_{\mu}\, p_{\nu} + \frac{1}{2}\,\Gamma(\alpha-3)\,\eta_{\mu\nu}\,(m^2-p^2)\,\} \qquad (5.29)$$

$$(\mathrm{Re}\,\alpha > 3)$$

Then

$$\tilde{\Lambda}^{\mu abc^{(A)}}(k_1,k_2,q;z)\Big|_{k_1^2=k_2^2=q^2=-\mu^2}$$

$$\simeq \frac{-\alpha\, g^3\, C_2(G) f^{abc}}{32\,\pi^2}\;\frac{\Gamma(1-z)}{\Gamma(z+1)}\left(\frac{\kappa^2\,\mu^2}{16\,\pi^2}\right)^z k_2^{\delta}$$

$$\int_0^1 dx \int_0^{1-x} dy\,(1-x-y)^{-z}\,\{[\{(k_2x+k_1y)^{\mu}\,(k_2x+k_1y)_{\delta}$$

$$-\,\eta^{\mu}_{\delta}(k_2x+k_1y).(k_2x+k_1y)\}\,\frac{\{x(1-x)+y(1-y)-xy\}^{z-1}}{\mu^2}\,]_{\mathrm{I}}$$

$$+[\,\frac{3}{2z}\,\eta^{\mu}_{\delta}\{x(1-x)+y(1-y)-xy\}^{z}]_{\mathrm{II}}\,\} \qquad (5.30)$$

Term I is well-defined at $z = 0$ and will not contribute to the double pole term in $\tilde{\Lambda}^{\mu abc}(k_1,k_2, q)$; hence it will not contribute to $O(\log \kappa)$. Therefore,

$$\tilde{\Lambda}^{\mu abc^{(A)}}(k_1,k_2,q;z)\Big|_{k_1^2=k_2^2=q^2=-\mu^2} \simeq \frac{-3\alpha g^3 C_2(G) f^{abc}}{64\pi^2}$$

$$\frac{\Gamma(1-z)}{z\Gamma(z+1)} \left(\frac{\kappa^2\mu^2}{16\pi^2}\right)^z k_2^\mu \int_0^1 dx \int_0^{1-x} dy(1-x-y)^{-z}$$

$$[x(1-x)+y(1-y)-xy]^z \qquad (5.31)$$

By similar considerations,

$$\tilde{\Lambda}^{\mu abc^{(B)}}(k_1,k_2,q;z)\Big|_{k_1^2=k_2^2=q^2=-\mu^2}$$

$$\simeq \frac{-\alpha g^3 C_2(G) f^{abc}}{64\pi^2} \frac{\Gamma(1-z)}{z\Gamma(z+1)} \left(\frac{\kappa^2\mu^2}{16\pi^2}\right)^z k_2^\mu \int_0^1 dx \int_0^{1-x} dy(1-x-y)^{-z}$$

$$[x(1-x) + y(1-y) - xy]^z \qquad (5.32)$$

Substituting eqns.(5.31) and (5.32) in (5.24) and evaluating the residue at $z = 0$,

$$\tilde{\Lambda}^{\mu abc}(k_1,k_2,q)\Big|_{k_1^2=k_2^2=q^2=-\mu^2}$$

$$= \frac{-\alpha g^3 C_2(G) f^{abc}}{16\pi^2} k_2^\mu\left[\log\left(\frac{\kappa\mu}{4\pi}\right) + O(\kappa^0)\right] \qquad (5.33)$$

Now $\tilde{Z}_1$, the Ghost-Ghost-vector renormalization constant is defined through the relation,

$$\tilde{\Gamma}^{\mu abc}(k_1,k_2,q)\Big|_{k_1^2=k_2^2=q^2=-\mu^2} = g k_2^\mu f^{abc} + \tilde{\Lambda}^{\mu abc}(k_1,k_2,q)\Big|\ldots$$

$$=(\tilde{Z}_1)^{-1} g k_2^\mu f^{abc} \qquad (5.34)$$

We then obtain,

$$\tilde{Z}_1 = 1 - \frac{\alpha g^2 C_2(G)}{16\pi^2} \left[\log\left(\frac{4\pi}{\kappa\mu}\right) + O(\kappa^0) \right] \qquad (5.35)$$

5.1(d) Vector-Vector-Vector Vertex

The relevant lowest order supergraphs are shown in Fig. 17.

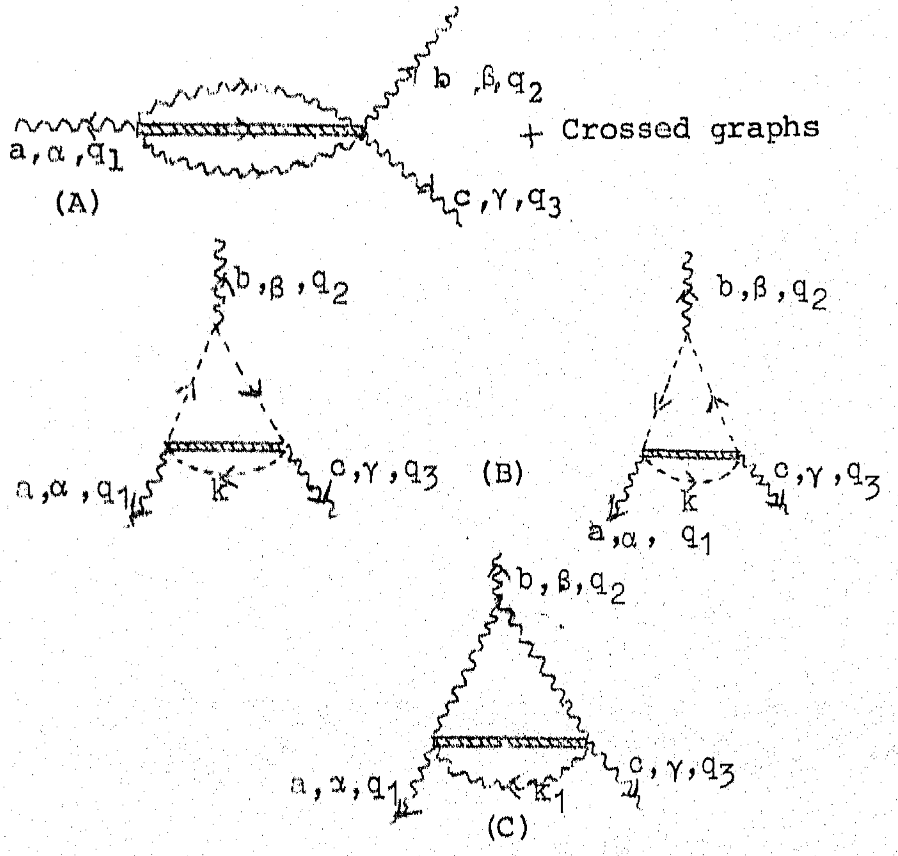

Fig.17 Corrections to the Vector-Vector-Vector Vertex.

We combine the superpropagator with the ghost propagator in Fig.17(B) and with the vector propagator in Fig.17(A) and Fig.17(C) using eqns. (5.22) and (5.23). There is a factor of ½ for Fig.17(A) due to combinatorics and a factor of (-1) in Fig.17(B) as there is a closed ghost loop. In Fig.17(B),

$$\Lambda^{abc^{C}}_{\alpha\beta\gamma}(q_1,q_2,q_3;z)$$

$$= \frac{i}{2} g^3 C_2(G) f^{abc} \frac{\Gamma(1-z)}{\Gamma(z+1)} \left(\frac{-\kappa^2}{16\pi^2}\right)^z$$

$$\int \frac{d^4k}{(2\pi)^4} [\{\eta_{\mu\alpha}(-k_1-q_1)_{\mu'} + \eta_{\alpha\mu'}(q_1-k_2)_{\mu} + \eta_{\mu\mu'}(k_1+k_2)_{\alpha}\}$$

$$\{\eta_{\nu\beta}(-k_2-q_2)_{\nu'} + \eta_{\beta\nu'}(q_2-k_3)_{\nu} + \eta_{\nu\nu'}(k_2+k_3)_{\beta}\}$$

$$\{\eta_{\lambda\gamma}(-k_3-q_3)_{\lambda'} + \eta_{\gamma\lambda'}(q_3-k_1)_{\lambda} + \eta_{\lambda\lambda'}(k_3+k_1)_{\gamma}\}]$$

$$(\eta^{\mu'\nu} - \alpha' \frac{k_2^{\mu'} k_2^{\nu}}{k_2^2})(\eta^{\nu'\lambda} - \alpha' \frac{k_3^{\nu'} k_3^{\lambda}}{k_3^2})$$

$$\{(1 - \frac{z}{2}\frac{\alpha'}{(z+1)})\eta^{\lambda'\mu} + \alpha' \frac{(z-1)}{(z+1)} \frac{k_1^{\lambda'} k_1^{\mu}}{k_1^2}\}$$

$$\frac{1}{(k_1^2)^{1-z}} \frac{1}{k_2^2} \frac{1}{k_3^2} \tag{5.40a}$$

where

$$k_1 = k + \frac{1}{3}(q_1-q_3), \quad k_2 = k + \frac{1}{3}(q_2-q_1), \quad k_3 = k + \frac{1}{3}(q_3-q_2) \tag{5.40b}$$

The rest of the calculation proceeds along the same lines as in the previous subsection. We obtain

$$\Lambda^{abc(A,B,C)}_{\alpha\beta\gamma}(q_1,q_2,q_3)\Big|_{q_1^2=q_2^2=q_3^2=-\mu^2}$$

$$= \frac{g^3 C_2(G)}{16\pi^2} f^{abc} \log(\frac{\kappa\mu}{4\pi})[\eta_{\alpha\beta}(q_1-q_2)_{\gamma} + \eta_{\beta\gamma}(q_2-q_3)_{\alpha}$$

$$+ \eta_{\gamma\alpha}(q_3-q_1)_{\beta}]\chi^{(A,B,C)} + O(\kappa^0) \tag{5.41a}$$

where

$$\chi^A = -\left(\frac{9}{2} - \frac{3}{4}\alpha'\right) = -\frac{15}{4} - \frac{3}{4}\alpha$$

$$\chi^B = -\frac{1}{12} \qquad (5.41b)$$

$$\chi^C = \frac{13}{4} - \frac{9}{4}\alpha' = 1 + \frac{9}{4}\alpha$$

Substituting these in eqn. (5.36) and defining Z_1 through the relation

$$\Lambda^{abc}_{\alpha\beta\gamma}(q_1,q_2,q_3)\Big|_{q_1^2=q_2^2=q_3^2=-\mu^2}$$

$$= (Z_1^{-1}-1)(-g)f^{abc}\left[\eta_{\alpha\beta}(q_1-q_2)_\gamma + \eta_{\beta\gamma}(q_2-q_3)_\alpha + \eta_{\gamma\alpha}(q_3-q_1)_\beta\right]$$

We obtain,

$$Z_1 = 1 + \frac{g^2C_2(G)}{16\pi^2}\left(\frac{17}{6} - \frac{3}{2}\alpha\right)\log\left(\frac{4\pi}{\kappa\mu}\right) + O(\kappa^0) \qquad (5.42)$$

The renormalization constants calculated above agree to order log κ with the conventional ones [43] with the ultraviolet cutoff Λ replaced by κ^{-1} and satisfy the Ward-identity,

$$\frac{Z_3}{Z_1} = \frac{\tilde{Z}_3}{\tilde{Z}_1} \qquad (5.43)$$

5.2 Gauge Invariance

In the above calculations, gauge invariance was not fully maintained. For example, in the calculated expression for $\Pi_{\mu\nu}$ in eqn. (5.12), there is a longitudinal part. It is known that **naive** perturbation theory may not respect all the symmetries in a Lagrangian; for example, in

conventional quantum electrodynamics, we get a non-gauge invariant result for photon self-energy [41] if we do not use a 'gauge-invariant' regularization [27]. Here, gauge invariance can be restored by replacing the electromagnetic current $\bar{\psi}\gamma^{\mu}\psi$ by the strictly gauge-invariant current operator,

$$\bar{\psi}\,(x+\tfrac{\varepsilon}{2})\;\gamma^{\mu}\psi(x-\tfrac{\varepsilon}{2})\exp\left[-ie\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} d\xi^{\nu}A_{\nu}(\xi)\right]$$

where the integral is along a space-like path [44]. This is called the'Point-splitting'method as the operators have different space-time arguments in the current. If we consider the additional interactions generated by this term and take the limit $\varepsilon\to 0$ only at the end, the amplitudes in higher orders will be gauge invariant [45]. The need for modification of the current operator is there in gravity-modified theories also, as the product of field operators at the same space-time point is singular as usual. In Refs. [10,11], it was shown that the gauge invariance of the 'scalar gravity' regularized quantum electrodynamics can be restored by employing the manifestly gauge-invariant current operator and including the effects of the discarded terms (the kinetic-energy modification terms)in the Lagrangian for convergence considerations.

Our discussion of gauge-invariance runs parallel to the discussion in Ref. [10]. The construction of a manifestly gauge-invariant current operator for the Yang-Mills

theory is not a trivial extension of the method for QED, due to the self-interactions of the Yang-Mills fields. Mandelstam [46] has proposed a gauge-independent, path-dependent, formalism for gauge theories. Using this approach, gauge covariant field variables have been constructed for Yang-Mills theory [47] and an explicit expression for the modified Lagrangian has also been obtained [48]. We follow the notation of Ref.[48].

Define an operator $\underset{\sim}{Y}(x)$ by:

$$\partial_\mu \underset{\sim}{Y}(x) = \frac{ig}{2} \underset{\sim}{Y}(x) \underset{\sim}{A}_\mu \qquad (5.44)$$

where $\underset{\sim}{A}_\mu = A^a_\mu t^a$

Here A^a_μ are the gauge fields and t^a are matrices in the adjoint representation of the generators of Yang-Mills group, satisfying

$$[t^a, t^b] = i f^{abc} t^c$$

and $\mathrm{Tr}\,[t^a t^b] = \delta^{ab}$

A solution of eqn. (5.44) is

$$\underset{\sim}{Y}(x) = T[\exp(\frac{ig}{2} \int_{-\infty}^{x} \underset{\sim}{A}_\mu(\xi) d\xi^\mu)] \qquad (5.45)$$

where T denotes ξ-ordering of the fields: the matrices $\underset{\sim}{A}_\mu(\xi)$ with larger values of ξ stand to the left of those with smaller values of ξ. Then the extension of the Point-splitting method to Yang-Mills theory consists [48] in replacing in the Lagrangian:

$$\underset{\sim}{F}_{\mu\nu} = t^a F^a_{\mu\nu} = \frac{1}{2}(D_\mu \underset{\sim}{A}_\nu - D_\nu \underset{\sim}{A}_\mu) + \text{ h.c.}$$

where $D_\mu = \partial_\mu - \frac{ig}{2} \underset{\sim}{A}_\mu$ by

$$\underset{\sim}{F}'_{\mu\nu} = \frac{1}{2} Y^{-1}(x, x-\varepsilon)\, D_\mu (x-\varepsilon)\{Y(x, x-\varepsilon)\underset{\sim}{A}_\mu (x)\}$$

$$- \frac{1}{2} Y^{-1}(x, x-\varepsilon) D_\nu (x-\varepsilon)\{\underset{\sim}{Y}(x, x-\varepsilon)\underset{\sim}{A}_\mu (x)\}+\text{h.c.} \qquad (5.46)$$

$$\text{where } Y(x,x-\varepsilon) = Y^{-1}(x-\varepsilon)Y(x) = 1-\frac{g}{2}\varepsilon^\lambda \underset{\sim}{A}_\lambda (x-\frac{\varepsilon}{2})+O(\varepsilon^2) \qquad (5.47)$$

Here h.c. stands for Hermitian conjugate. Taking into account the extra ε-dependent terms and going to the limit $\varepsilon \to 0$ in the end, it has been shown [48] that in the radiation gauge, the polarization tensor has the correct structure in the one-loop approximation; specifically the gauge field will not acquire mass. In a general, Lorentz covariant gauge also, one would obtain gauge invariant results in this approach, as gauge invariance is preserved at every step in Mandelstam's approach. Writing

$$\Pi_{\mu\nu}(k^2, \varepsilon) = (k_\mu k_\nu - \eta_{\mu\nu}k^2)C(k^2,\varepsilon) + \eta_{\mu\nu}D(k^2,\varepsilon),$$

$$\lim_{\varepsilon^\nu \to 0} D(k^2, \varepsilon) = 0 \qquad (5.48)$$

Here it should be noted that the Lagrangian should not be normal-ordered.

In the gravity-modified theory with the Yang-Mills part of the Lagrangian treated as above, we can write

$$[D(k^2)]_{total} = \frac{1}{2\pi i} \int_{C'_0} dz\, \Gamma(-z) D(k^2, z)$$

$$= D(k^2, o) + \frac{1}{2\pi i} \int_{C'_1} dz\, \Gamma(-z)(\kappa^2)^z D(k^2, z) \qquad (5.49)$$

where the contour C'_1 lies parallel to the imaginary axis with $0 < \mathrm{Re} z < 1$. $D(k^2, o)$ is what we would have got in the theory without including gravity. Considering it as $\lim_{\varepsilon_\nu \to 0} D(k^2, \varepsilon)$, eqn.(5.48) gives

$$D(k^2, o) = 0$$

Hence

$$[D(k^2)]_{total} = \frac{1}{2\pi i} \int_{C_1} dz\, \Gamma(-z)(\kappa^2)^z D(k2, z) \qquad (5.50)$$

Had the quantity on the right been finite as such, the demonstration of gauge invariance to $O(\kappa^0)$ would have been complete. This is not so. Consider the contribution to $D(k^2)$ from Fig.14(a) for example. For simplicity, we work in the Feynman gauge hereafter ($\alpha = 1$).

For this,

$$D(k^2, z)^{(a)} \sim \frac{\Gamma(2-z)}{\Gamma(z)} \int_{C'_1} d^4p\, d^4q \frac{[ap^2 + bq^2 + cp.q + \ldots]}{p^2 (k-p-q)^2} (q^2)^{z-2} \qquad (5.51)$$

This is convergent only for $\mathrm{Re} z < -1$. Hence the contour C'_1 would have to be shifted to the left. Following [10], we will now show that the inclusion of $L_{K.M.}$ in eqn.(5.5), the

kinetic energy modification term which formally gives contributions of $O(\kappa)$, makes the right hand side of eqn. (5.51) convergent without shifting the contour C_1' to the left.

To include the effects of $L_{K.M.}$, it is sufficient to consider the diagram shown in Fig.18.

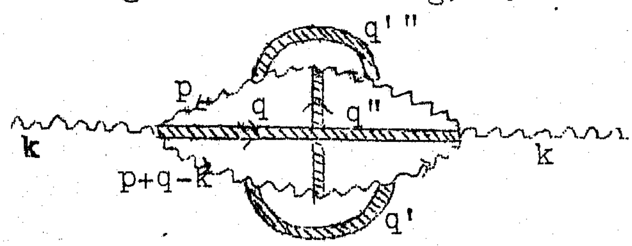

Fig.18 A Modified Gauge-Particle Self-Energy Supergraph.

Here the supergraph represents the sum of diagrams with and without modifications due to $L_{K.M.}$. 'Kinking' and 'Cradling' mentioned in Chapter IV is understood.

It is useful to consider the modified vector-vector-graviton supervertex shown in Fig. 19.

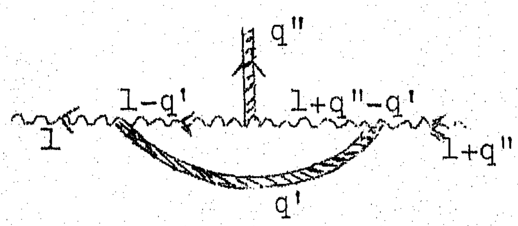

Fig.19 Vector-Vector-Graviton Supervertex.

The matrix elements for this diagram is:

$$M(l,q'')\sim\int_{\alpha-i\infty}^{\alpha+i\infty} dz_1\,(\kappa^2)^{z_1}\,h(z_1)\int d^4q'\;l.(l-q')\,\frac{1}{(l-q')^2}$$
$$\cdot(l-q').(l+q''-q')\frac{1}{(l+q''-q')^2}(l+q''-q').(l+q'')(q'^2)^{z-2} \tag{5.52}$$

where $-1<\alpha<0$. The function $h(z\ )$ includes the z-dependent

factors other than $(q^2)^{z-2}$ and various constants. The integral is convergent for $-1 < \alpha < 0$. Next, we examine the high-momentum behaviour of $M(l,q'')$.

Putting $|q''| = s^2$, we can show that

$$\lim_{s\to\infty} M(l,s^2,\hat{q}'') \lesssim s^{-1}(\log s)^{n_1} \qquad (5.53)$$

for $-1 < \alpha < -\frac{3}{4}$. Here, n_1 is a non-negative integer. Similarly, putting $|l| = t^2$,

$$\lim_{t\to\infty} M(t^2,\hat{l}\,,\,q'') \lesssim t(\log t)^{n_2} \qquad (5.54)$$

for $-1< \alpha < -\frac{3}{4}$. Finally, we find the behaviour of $M(l,q'')$ when both q'' and l tend to infinity.

Set $q'' = \omega^2\,\hat{q}''$, $l = a\,\omega^2\hat{l}$ where a is a positive number. Then

$$\lim_{\omega\to\infty} M(\omega\,,\,a\,\hat{l},\,\hat{q}'') \lesssim \omega(\log\omega)^{n_3},\ \text{when } -1<\alpha<-\frac{3}{4}. \qquad (5.55)$$

Now

$$D(k^2)^{(a)} \sim \int_{\beta-i\infty}^{\beta+i\infty} dz\,h(z)(\kappa^2)^z \int_{\alpha-i\infty}^{\alpha+i\infty} dz'\,h(z')\,(\kappa^2)^{z'}$$

$$\int d^4p\,d^4q''\,d^4q\{a_1p+b_1(p+q-k)\}^{\lambda}\frac{1}{p^2}\,M(p,-q'')\,\frac{1}{(p+q'')^2}$$

$$\frac{1}{(p+q-k)^2}\,M(p+q-q''-k,q'')\,\frac{1}{(p-q''+q-k)^2}\{a_2(p+q'')+b_2(p-q''+q-k)\}_{\lambda}$$

$$(q^2)^{z-2}(q''^2)^{z'-2},\ (-1<\alpha<0;\ 0<\beta<1) \qquad (5.56)$$

where a_1,a_2,b_1,b_2 are some fixed numbers. By power counting,

we can see that q", q and p integrations are all convergent, provided,

$$-1 < \alpha < -\frac{3}{4} \quad \text{and} \quad 0 < \beta < \frac{3}{4} .$$

No shifting of contours is involved. Collapsing the contours to lie parallel to the real axis, we can see that

$$D(k^2)^{(a)} \lesssim O(\kappa^2 \log \kappa) \tag{5.57}$$

Same arguments can be applied to other diagrams also. Hence, $\Pi_{\mu\nu}$ is transverse upto $O(\kappa^0)$.

By a similar procedure, gauge invariance is expected to be restored in other amplitudes also.

<u>Gravitational gauge-invariance</u>

Consider a general gauge for the gravitational field in which eqn.(3.7) is replaced by

$$(0\,|T(\,\phi^{\mu a}(x)\,\phi^{\mu b}(0))|\,0) = \frac{1}{2}(\,\eta^{\mu\nu}\eta^{ab} + \eta^{\mu b}\eta^{\nu a} - \xi\eta^{\mu a}\eta^{\nu b})D(x) \tag{5.58}$$

Then

$$(0|T(\chi(x)\chi(0))\,|\,0) = \frac{1}{4}(0\,|T(\mathrm{Tr}\,\phi(x)\mathrm{Tr}\phi(0))|\,0) = (1-2\xi)D(x) \tag{5.59}$$

Hence, we would have

$$(0\,|T(:e^{-\kappa\chi(x)}:\ :e^{-\kappa\chi(0)}:)|0) = e^{-\kappa^2(2\xi-1)D(x)} \tag{5.60}$$

Comparing this with eqn. (3.13), we see that κ^2 will be replaced by $\kappa^2(2\xi-1)$ in all the calculations. To $O(\kappa^0)$, only the renormalization constants would be dependent on ξ . Hence, in

this order, the renormalized amplitudes will be independent of the gravitational gauge-parameter, ξ.

5.3 Renormalization Constants in Quantum Chromodynamics

In this section, we present the calculation of renormalization constants in quantum chromodynamics [49], the theory of strong interactions based on unbroken SU(3) (colour) gauge invariant interactions of fermionic quarks and octet gluons. The Lagrangian in L of eqn.(5.2) (where now $a=1, \ldots, 8$) will be supplemented by the quark term (for simplicity, we will consider only one quark flavour),

$$L_{total} = \sqrt{(-g)}\,[\tfrac{1}{2}i(\bar{\psi}\gamma_\alpha\psi_{;\mu} - \bar{\psi}_{;\mu}\gamma_\alpha\psi)L^{\mu\alpha} - m\bar{\psi}\psi$$
$$+ g\bar{\psi}\gamma_\alpha A^a_\mu T_a\psi L^{\mu\alpha}] + L(5.2) \qquad (5.61)$$

where T_a are the matrices representing the generators of SU(3) in the representation to which quarks belong. In our approximation, $\psi_{;\mu}$ will be replaced by the ordinary derivative $\partial_\mu\psi$ and $L^{\mu\alpha}$ by $\eta^{\mu\alpha}$.

The renormalization constants $\tilde{Z}_3$ and $\tilde{Z}_1$ are the same as given in eqns. (5.21) and (5.35) in Section 5.1 with $C_2(G) = 3$. The relevant details for other renormalization constants are given below.

5.3(a) Gluon self-energy

The gluon (gauge particle) self-energy has an additional contribution from the fermion exchange shown in Fig.20, which we call $\Pi^F_{\mu\nu}$.

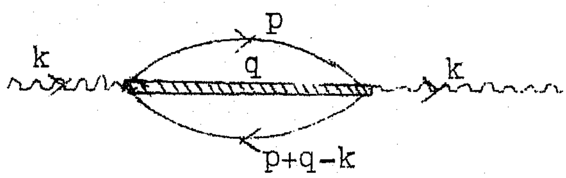

Fig.20 Fermion Contribution to Gluon Self-Energy.

$$\Pi^{F}_{\mu\nu}(k) = \frac{1}{2\pi i} \int_{C_o} dz\, \Gamma(-z)\, \Pi^{F}_{\mu\nu}(k,z) \tag{5.62}$$

where

$$\Pi^{F}_{\mu\nu}(k,z) = -g^2 T(R)(\kappa^2)^z (4\pi)^{2-2z} \frac{\Gamma(2-z)}{\Gamma(z)}$$

$$\int \frac{d^4p\, d^4q}{(2\pi)^8} \mathrm{Tr}\,[\gamma_\mu \frac{(\gamma.p+m)}{p^2-m^2} \gamma_\nu \frac{\gamma.(p+q-k)+m}{(p+q-k)^2-m^2}] (-q^2)^{z-2} \tag{5.63}$$

Here $T(R) = \mathrm{Tr}\,(T_a)^2$; for triplets, $T(R) = \frac{1}{2}$.

Except for the factor $T(R)$, the expression for $\Pi^F_{\mu\nu}$ is identical with the photon self-energy in Ref.[10]. Writing,

$$\Pi^F_{\mu\nu} = (k^2\eta_{\mu\nu} - k_\mu k_\nu)\, C(k^2) + \eta_{\mu\nu} D(k^2) \tag{5.64}$$

We would have,

$$C(k^2) = \frac{g^2 T(R)}{12\pi^2}[\log(\frac{4\pi^2}{m^2\kappa^2}) + O(\kappa^o)]$$

and

$$D(k^2) = \frac{2}{\kappa^2} g^2 T(R) + O(\log \kappa) \tag{5.65}$$

Again there is a longitudinal part in $\Pi^F_{\mu\nu}$ which can be remedied by the procedure discussed in Section 5.2, to ensure gauge invariance. (The gauge covariant operator for the fermionic current using the point-splitting method in non-Abelian gauge theories is discussed in Ref.[50]).

The fermionic contribution to Z_3 is given by

$$(Z_3-1)^F = -C(0)$$
$$= -\frac{g^2 T(R)}{6\pi^2}\,[\,\log(\frac{2\pi}{m\kappa}) + O(\kappa^0)\,] \qquad (5.66)$$

From eqns. (5.15) and (5.66),

$$Z_3 = 1 + \frac{g^2}{16\pi^2} C_2(G)\,[\,(\frac{13}{3}-\alpha)\log(\frac{4\pi}{\kappa\mu}) + O(\kappa^0)\,]$$
$$- \frac{g^2}{6\pi^2} T(R)\,[\,\log(\frac{2\pi}{m\kappa}) + O(\kappa^0)\,] \qquad (5.67)$$

5.3(b) Vector-Vector-Vector Vertex

Additional supergraphs shown in Fig. 21 the sum of which we term $\Lambda^{abc\,(F)}_{\alpha\beta\gamma}$ contribute to $\Lambda^{abc}_{\alpha\beta\gamma}$.

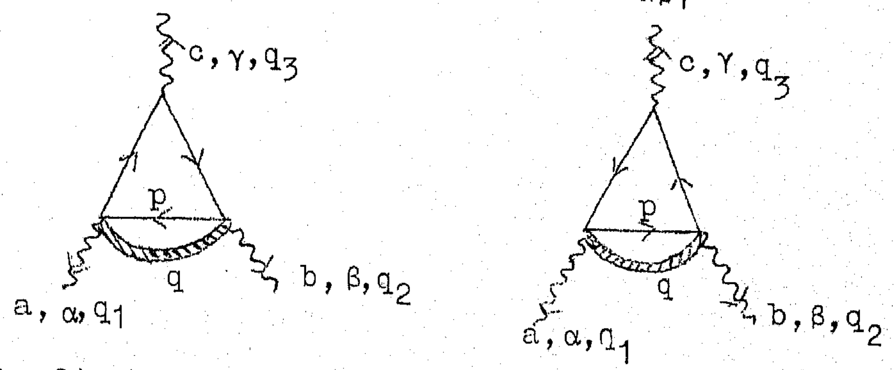

Fig.21 Fermionic Contribution to V-V-V Vertex Correction.

We have

$$\Lambda^{abc^F}_{\alpha\beta\gamma}(q_1,q_2,q_3) = \frac{1}{2\pi i}\int_{C_0} dz\ \Gamma(-z)\,\Lambda^{abc^F}_{\alpha\beta\gamma}(q_1,q_2,q_3;z) \qquad (5.68)$$

where

$$\Lambda^{abc^F}_{\alpha\beta\gamma}(q_1,q_2,q_3;z) = \frac{g^3}{2} T(R) f^{abc} (\kappa^2)^z \frac{\Gamma(2-z)}{\Gamma(z)} (4\pi)^{2-2z}$$
$$\int \frac{d^4p\,d^4q}{(2\pi)^8} \mathrm{Tr}\,[\,\gamma_\alpha \frac{1}{\not p - m}\gamma_\beta \frac{1}{\not p+\not q+\not q_2 - m}\gamma_\gamma \frac{1}{\not p+\not q-\not q_1 - m}$$
$$- \gamma_\alpha \frac{1}{\not p+\not q+\not q_1 - m}\gamma_\gamma \frac{1}{\not p+\not q-\not q_2 - m}\gamma_\beta \frac{1}{\not p - m}\,]\,(-q^2)^{z-2} \qquad (5.69)$$

We can use the relation satisfied by Dirac-matrices,

$$\frac{1}{\not p-m}(\not p-\not q)\frac{1}{\not q-m} = \frac{1}{\not q-m} - \frac{1}{\not p-m} \qquad (5.70)$$

to obtain

$$q_3^{\gamma} \quad \Lambda^{abc^F}_{\alpha\beta\gamma}(q_1,q_2,q_3;z) = -(q_1+q_2)^{\gamma} \quad \Lambda^{abc^F}_{\alpha\beta\gamma}(q_1,q_2,q_3;z)$$

$$= -g\ [\ \Pi^F_{\alpha\beta}(q_2,z) - \Pi^F_{\alpha\beta}(q_1,z)\ ] \qquad (5.71)$$

where $\Pi^F_{\alpha\beta}(k,z)$ is given by eqn. (5.63). Now it is straightforward to compute $(Z_1-1)^F$. We get,

$$(Z_1-1)^F = \frac{-g^2}{6\pi^2} T(R)\ [\ \log(\frac{2\pi}{m\kappa}) + O(\kappa^0)\ ] \qquad (5.72)$$

$$= (Z_3-1)^F.$$

From (5.72) and (5.42),

$$Z_1 = 1 + \frac{g^2 C_2(G)}{16\pi^2}\left(\frac{17}{6} - \frac{3}{2}\alpha\right)[\log(\frac{4\pi}{\kappa\mu}) + O(\kappa^0)\ ]$$

$$- \frac{g^2 T(R)}{6\pi^2}[\log(\frac{2\pi}{m\kappa}) + O(\kappa^0).]. \qquad (5.73)$$

5.3(c) <u>Fermion self-energy</u>

The Fermion self-energy supergraph is shown in Fig. 22.

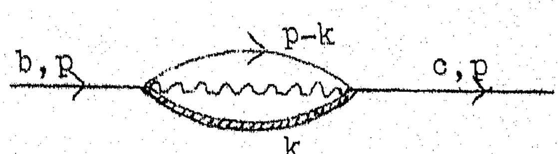

Fig. 22 Fermion Self-Energy Supergraph.

It is convenient to combine the vector propagator $D^{ab}_{\mu\nu}(x)$ and the superpropagator $\mathcal{D}(x)$ using eqn. (5.23). Then, we obtain

$$\Sigma(p)_{bc} = \delta_{bc}\ (\not p\ A(p^2) + mB(p^2)) = \frac{\delta_{bc}}{2\pi i}\int_{C_0} dz\ \Gamma(-z)\ \Sigma(p,z) \qquad (5.74)$$

$$\Sigma(p,z) = \not{p}\, A(p^2,z) + m\, B(p^2,z)$$

$$= -ig^2 C_2(R) \frac{\Gamma(1-z)}{\Gamma(1+z)} \left(\frac{-\kappa^2}{16\pi^2}\right)^z$$

$$\int \frac{d^4k}{(2\pi)^4} \gamma_\mu \frac{(\not{p}-\not{k}+m)}{(p-k)^2-m^2} \gamma_\nu \left\{ \left(1 - \frac{z\alpha'}{2(z+1)}\right) \eta^{\mu\nu} + \alpha' \frac{k^\mu k^\nu}{k^2} \right\} (k^2)^{z-1} \tag{5.75}$$

Here $C_2(R)$ is the quadratic Casimir invariant in the representation of fermions, defined by

$$\sum_a (T^a)^2_{bc} = C_2(R)\, \delta_{bc} \tag{5.76}$$

In the triplet representation of SU(3), $C_2(R) = 4/3$.

$$A(p^2,z) = \frac{1}{4p^2} \mathrm{Tr}(\not{p}\, \Sigma(p,z))$$

$$= -ig^2 C_2(R) \frac{\Gamma(1-z)}{\Gamma(1+z)} \left(\frac{-\kappa^2}{16\pi^2}\right)^z \frac{1}{p^2}$$

$$\int \frac{d^4k}{(2\pi)^4} \{ 2p_\mu(p_\nu - k_\nu) - p.(p-k)\, \eta_{\mu\nu} \} \{ \left(1 - \frac{z\alpha'}{2(z+1)}\right) \eta^{\mu\nu}$$

$$+ \frac{\alpha'(z-1)}{(z+1)} \frac{k^\mu k^\nu}{k^2} \} [(p-k)^2 - m^2]^{-1} (k^2)^{z-1} \tag{5.77}$$

$$B(p^2,z) = ig^2 C_2(R) \frac{\Gamma(1-z)}{\Gamma(1+z)} \left(\frac{-\kappa^2}{16\pi^2}\right)^z (4-\alpha')$$

$$\int \frac{d^4k}{(2\pi)^4} [(p-k)^2 - m^2]^{-1} (k^2)^{z-1} \tag{5.78}$$

$A(p^2,z)$ and $B(p^2,z)$ can be evaluated by the methods described in Section 5.1(c). Finally, we obtain:

$$A(p^2) = \frac{g^2}{16\pi^2} C_2(R)\, \alpha \left[\log\left(\frac{\kappa^2 m^2}{16\pi^2}\right) + O(\kappa^0) \right] \tag{5.79}$$

$$B(p^2) = \frac{-g^2}{16\pi^2} C_2(R)\ (3+\alpha)\left[\log\left(\frac{\kappa^2 m^2}{16\pi^2}\right) + O(\kappa^0)\right] \tag{5.80}$$

The fermion wave function renormalization constant is given to $O(\log \kappa)$ by

$$Z_2 = (1-A(m^2))^{-1} = 1 - \frac{\alpha g^2 C_2(R)}{8\pi^2}\left[\log\left(\frac{4\kappa}{\kappa m}\right) + O(\kappa^0)\right] \tag{5.81}$$

The fermion self-mass is

$$\begin{aligned}\delta m &= m\left[A(m^2) + B(m^2)\right] \\ &= 3m\,\frac{g^2 C_2(R)}{8\pi^2}\left[\log\left(\frac{4\pi}{\kappa m}\right) + O(\kappa^0)\right]\end{aligned} \tag{5.82}$$

5.3(d) Fermion-Fermion-Vector Vertex

For the fermion-fermion-vector vertex, the relevant supergraphs are shown in Fig. 23.

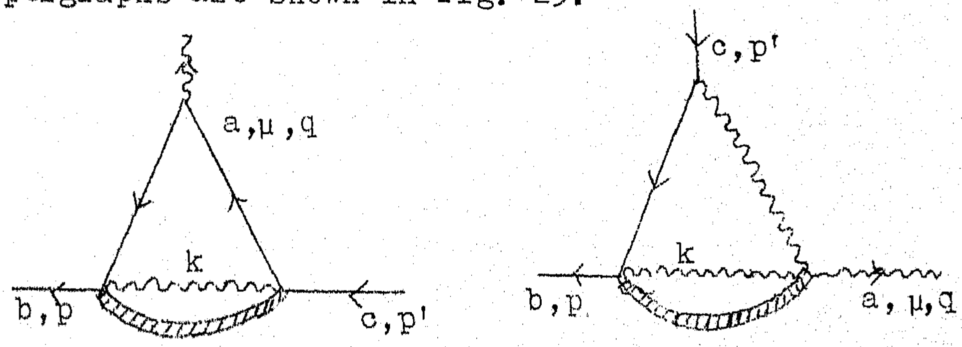

Fig.23 Fermion-Fermion-Vector Vertex Supergraphs.

As before, we have kept a superpropagator only between one pair of vertices. It is convenient to keep the superpropagator running parallel to a gluon line because it facilitates the application of the calculus of derivatives. Then,

$$\Lambda^{abc^{\mu}}_{FFV}(q;\ p,p') = \Lambda^{abc^{\mu(A)}}_{FFV} + \Lambda^{abc^{\mu(B)}}_{FFV} \tag{5.83}$$

where

$$\Lambda^{abc\ \mu(A)}_{FFV}(q;p,p') = g^3\,[C_2(R)-\tfrac{1}{2}C_2(G)]\,(T^a)_{bc}\,\frac{1}{2\pi i}\int_{C_0} dz$$

$$\frac{\Gamma(-z)\Gamma(1-z)}{\Gamma(1+z)}\left(\frac{-\kappa^2}{16\pi^2}\right)^z \int \frac{d^4k}{(2\pi)^4}\,\gamma^\nu\,\frac{1}{(\not p+\not k-m)}\,\gamma^\mu\,\frac{1}{(\not p'+\not k-m)}\,\gamma^\lambda$$

$$\left\{\left(1-\frac{z\alpha'}{2(z+1)}\right)\eta_{\nu\lambda}+\alpha'\frac{(z-1)}{(z+1)}\,\frac{k_\nu k_\lambda}{k^2}\right\}(k^2)^{z-1} \qquad (5.84)$$

and

$$\Lambda^{abc\ \mu(B)}_{FFV}(q;p,p') = \frac{-g^3}{2}\,C_2(G)(T^a)_{bc}\,\frac{1}{2\pi i}\int_{C_0} dz\,\frac{\Gamma(-z)\Gamma(1-z)}{\Gamma(z+1)}$$

$$\left(\frac{-\kappa^2}{16\pi^2}\right)^z\int\frac{d^4k}{(2\pi)^4}\left[\gamma^{\lambda'}\,\frac{(\not p+\not k+m)}{(p+k)^2-m^2}\,\gamma^{\nu'}\{\eta^{\mu\nu}(2q-k)^\lambda+\eta^{\nu\lambda}(2k-q)^\mu\right.$$

$$+\ \eta^{\lambda\mu}(-k-q)^\nu\}\left\{\eta_{\nu\nu'}-\alpha'\frac{(k-q)_\nu(k-q)_{\nu'}}{(k-q)^2}\right\}\frac{1}{(k-q)^2}$$

$$\left.\left\{\left(1-\frac{z}{2}\,\frac{\alpha'}{(z+1)}\right)\eta_{\lambda\lambda'}+\alpha'\frac{(z-1)}{(z+1)}\,\frac{k_\lambda k_{\lambda'}}{k^2}\right\}\right]\ (k^2)^{z-1} \qquad (5.85)$$

Using the relation (5.70), it is easy to show that:

$$(p'-p)_\mu\,\Lambda^{abc\ \mu(A)}_{FFV} = ig\,\frac{[C_2(R)-\frac{1}{2}C_2(G)]}{C_2(R)}\,(T^a)_{bc}\,[\Sigma(p)-\Sigma(p')] \qquad (5.86)$$

with $\Sigma(p) = \not p\,A(p^2)+mB(p^2)$ and $A(p^2)$ and $B(p^2)$ given by eqns.(5.79) and (5.80). Hence

$$\Lambda^{abc\ \mu(A)}_{FFV} = \frac{ig^3\,\alpha\,(T^a)_{bc}\,\gamma^\mu}{16\,\pi^2}\,[C_2(R)-\tfrac{1}{2}C_2(G)]\left[\log\left(\frac{\kappa^2 m^2}{16\pi^2}\right)+O(\kappa^0)\right] \qquad (5.87)$$

$\Lambda^{abc\ \mu(B)}_{FFV}$ can be evaluated by the method described in Section (5.1.c) yielding:

$$\Lambda^{abc\ \mu (B)}_{FFV} = -3i(1+\alpha)\frac{g^3 C_2(G)}{64\pi^2}(T^a)_{bc}\gamma^\mu[\log(\frac{\kappa^2 m^2}{16\pi^2}) + O(\kappa^0)] \tag{5.88}$$

Defining

$$\Lambda^{abc\ \mu}_{FFV\ unrenorm.}\Big|_{\not p=\not p'=m} = ig(T^a)_{bc}\ \gamma^\mu + \Lambda^{abc\ \mu}_{FFV}\Big|_{\not p=\not p'=m}$$

$$= (Z_1')^{-1}\ ig(T^a)_{bc}\gamma^\mu \tag{5.89}$$

We obtain,

$$Z_1' = 1 - \frac{g^2}{16\pi^2}\ \{2\alpha C_2(R) + \frac{1}{2}(\alpha + 3)C_2(G)\}\log(\frac{4\pi}{\kappa m}) + O(\kappa^0) \tag{5.90}$$

To lowest order in g, all the renormalization constants correspond to the usual ones [49] with the cutoff Λ replaced by κ^{-1}. To $O(\log\kappa)$, the following Ward-identities are satisfied:

$$\frac{Z_3}{Z_1} = \frac{\tilde{Z}_3}{\tilde{Z}_1} = \frac{Z_2}{Z_1'} \tag{5.91}$$

5.4 Spontaneously Broken Gauge Theories

Non-Abelian gauge theories with spontaneously symmetry breaking provide the basis for unified, renormalizable theories of weak, electromagnetic and strong interactions [51]. Spontaneously broken gauge theories (SBGT) have the features of both unbroken gauge theories and theories with spontaneous breaking which we have already considered separately. Using the methods employed there, SBGT can also be regularized.

through the inclusion of gravity. As any calculation including the graviton-superpropagators would be very involved, we confine ourselves to some general comments, regarding the regularization.

In SBGT, the unitary gauge is the one in which there are no unphysical degrees of freedom. It is not obvious that the Lagrangian is renormalizable in this gauge [52]. Also there are ambiguities in calculations performed in this gauge[53]. With the incorporation of gravity in our approximation, it is clear from the arguments of Section 3.2 that we get finite amplitudes in the manifestly non-renormalizable unitary gauge also. Also, the results would be unambiguous. Corresponding to the quartic divergences of the original theory, the regularized amplitudes would contain terms of $O(\kappa^{-4})$. But the renormalized amplitude would be the same as in the original theory with any regularization to $O(\kappa^{0})$.

The natural cutoff provided by gravity has some interesting implications for unified theories of weak, electromagnetic and strong interactions. Using renormalization group arguments, the disparities in the strengths of these interactions can be understood if we assume that at energies of the order $\kappa^{-1} \approx 10^{19}$ GeV, the coupling constants for these interactions are the same [54]. Using the same argument, Shafi [55] has shown that it places constraints on the possible models which are candidates for a unified theory of these interactions.

CHAPTER VI

AXIAL VECTOR ANOMALY

When fermions are present in a gauge theory, the axial vector Ward identity has an anomalous term [56].With our regularisation, we get an anomalous term which coincides with the standard one to $O(\kappa^{0})$.

In the first section, we give a brief introduction to the problem of anomalies. In the next section, the contribution of the triangle graph to the vector-vector-axial vector (VVA) vertex in gravity-modified quantum electrodynamics is evaluated and the anomalous term determined. Defining the currents carefully using the 'point-splitting method' [44]and the equations of motion, we get the same answer. This can be extended to non-Abelian gauge theories also.

6.1 Introduction

Consider the axial vector current in QED. (Details are given in Refs. [57,58]). From the Lagrangian,

$$L = i\bar{\Psi}\gamma^{\mu} \left(\partial_{\mu} - ieA_{\mu}\right)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - m \bar{\Psi}\psi \tag{6.1}$$

We have, for the vector current $j_{\mu}(x)$ and the axial vector current $j^{5}_{\mu}(x)$, the formal expressions:

$$j_{\mu}(x) = \bar{\Psi}\gamma_{\mu}\psi(x)$$

and

$$j^{5}_{\mu}(x) = \bar{\Psi}\gamma_{\mu}\gamma_{5}\, \psi(x) \tag{6.2}$$

Using the equations of motion, we get the 'naive' conservation equations,

$$\partial^{\mu} j_{\mu} = 0$$

$$\partial^{\mu} j^{5}_{\mu} = 2im\, \bar{\psi}\, \gamma_5\, \psi \equiv 2im\, j^{5}(x) \tag{6.3}$$

The vertex parts for the vector, axial vector and pseudo-scalar currents $\Gamma_{\mu}(p,p')$, $\Gamma^{5}_{\mu}(p,p')$ and $\Gamma^{5}(p,p')$ respectively are defined through the relations:

$$S'_F(p)\, \Gamma_{\mu}(p,p')\, S'_F(p') = - \int d^4x d^4y e^{i(p.x-p'.y)} (0|\, T(\, \psi(x) j_{\mu}(0) \bar{\psi}(y))|\, 0), \tag{6.4}$$

$$S'_F(p)\, \Gamma^{5}_{\mu}(p,p') S'_F(p') = - \int d^4x d^4y e^{i(p.x-p'.y)} (0|T(\psi(x) j^{5}_{\mu}(0)\, \bar{\psi}(y))|0) \tag{6.5}$$

and

$$S'_F(p)\, \Gamma^{5}(p,p')\, S'_F(p') = - \int d^4x d^4y\, e^{i(p.x-p'.y)} (0|\, T(\, \psi(x)\, j^{5}(0)\, \bar{\psi}(y))|0) \tag{6.6}$$

where $S'_F(p)$ is the full electron propagator.

From formal manipulations, one would get the vector Ward-identity,

$$(p-p')^{\mu}\, \Gamma_{\mu}(p,p') = S'_F(p)^{-1} - S'_F(p')^{-1} \tag{6.7}$$

and the axial vector Ward-identity,

$$(p-p')^{\mu}\, \Gamma^{5}_{\mu}(p,p') = 2m\, \Gamma^{5}(p,p') + S'_F(p)^{-1}\, \gamma_5 + \gamma_5\, S'_F(p')^{-1} \tag{6.8}$$

But these formal manipulations may not be actually valid in perturbation theory. The verification of these

Ward-identities involves shifts in momentum integration variables. But the Feynman graphs contributing to these vertex functions are linearly divergent and one knows that shifts in integration variables are not allowed when the divergences are linear or higher [41]. It has been shown that one cannot simultaneously satisfy both the vector and axial vector Ward identities. As an example, consider the triangle graph below, contributing to the VVA vertex (Fig.24).

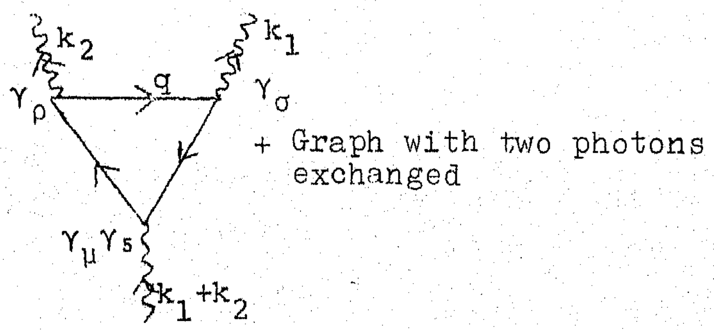

Fig.24 The Triangle Diagram.

We have for the contribution of this diagrams,

$$\frac{-ie^2}{(2\pi)^4} R_{\sigma\rho\mu}(k_1,k_2) = -2ie^2 \int \frac{d^4q}{(2\pi)^4} \mathrm{Tr}\left[\frac{1}{\not{q}+\not{k}_1-m}\gamma_\sigma \frac{1}{\not{q}-m}\right.$$

$$\left.\gamma_\rho \frac{1}{\not{q}-\not{k}_2-m}\gamma_\mu\gamma_5\right] \qquad (6.9)$$

This is a linearly divergent integral. From formal considerations, for this diagram, the vector and axial-vector Ward-identities, will have the form:

$$k_1^\sigma \; R_{\sigma\rho\mu} = k_2^\rho \; R_{\sigma\rho\mu} = 0 \qquad (6.10)$$

and $-(k_1+k_2)^\mu R_{\sigma\rho\mu} = 2m\, R_{\sigma\rho}$ (6.11)

where $R_{\sigma\rho}$ is the Feynman graph in Fig. 24 with $\gamma_\mu\gamma_5$ replaced by γ_5. But an explicit evaluation, using the vector-current conservation equation (eqn.6.10) gives the result [57]

$$-(k_1+k_2)^\mu \; R_{\sigma\rho\mu} = 2m \; R_{\sigma\rho} + 8\pi^2 \, \varepsilon_{\xi\tau\sigma\rho} \; k_1^\xi \;\; k_2^\tau \tag{6.12}$$

The additional term is the anomolous term. Equation (6.3) would then have to be modified to:

$$\partial^\mu j_\mu^5 = 2im \; j^5 + \frac{e^2}{16\pi^2} \varepsilon_{\mu\nu\rho\sigma} \;\; F^{\mu\nu} F^{\rho\sigma} \tag{6.13}$$

Equation (6.13) could also be obtained by a more careful, gauge-invariant definition of the current j_μ^5 by the 'point-splitting method'. We defer this to Section 6.3.

The question arises as to whether the result depends on the regularization procedure one adopts. In the dimensional regularization [59], the problem assumes a different form: objects like γ_5 and $\varepsilon_{\mu\nu\rho\sigma}$ have a meaning only for four-dimensional space-time. Delbourgo and Akeympong [59] have shown that by a suitable definition of objects involving γ_5 applicable to n-dimensions, the PCAC anomaly (anomaly in the equation $\partial_\mu \vec{A}^\mu = C \vec{\pi}$) would have the desired form. But this procedure has one source of trouble [60]: the vector current in weak interactions will then have one more term which will be renormalized by strong interactions violating CVC.

Recently, Kapoor [61] has shown that various regularizations and also regulator-free BPHZ approach

lead to the same anomaly for the AAA vertex in the theory of massless fermions coupled to a massive axial vector meson.

In the next section, we show that in gravity-modified quantum electrodynamics in our approximation, we have the same anomaly for the axial vector current equation as in eqn. (6.13) to $O(\kappa^0)$, by considering the triangle graph.

6.2 Axial-Vector Anomaly in Gravity-Modified QED

The interaction Lagrangian for gravity-modified QED in our approximation is given by eqns. (3.10) to (3.12). We will not consider $L_{K.M.}$ which is formally of $O(\kappa)$, in the following. Then

$$L'_{em} = e\, e^{-\kappa\chi}\, \bar{\Psi}\gamma_\mu\psi\, A^\mu$$

For simplicity, we set m, the mass of the electron equal to zero.

The triangle graph contributing to the VVA vertex is now modified by the inclusion of the superpropagator. As before, we have included one superpropagator only (Fig. 25).

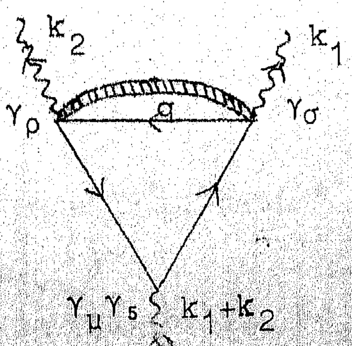

Fig. 25 Triangle Diagram with the Superpropagator.

We combine the electron propagator between the photon lines and the superpropagator, using the calculus of derivatives and Gelfand-Shilov formula:

$$F[\,(i\gamma^{\mu}\partial_{\mu})\, D(x)e^{-\kappa^2 D}\,](q) \equiv i\gamma^{\mu}\,\frac{1}{2\pi i}\int_{C_0} dz\,\frac{\Gamma(-z)}{\Gamma(z+1)}\,(\kappa^2)^z \; F[\,\partial_{\mu}\, D^{z+1}\,](q)$$

$$= \not{q}\;\frac{1}{16\,\pi^2 i}\;\frac{1}{2\pi i}\int_{C_0} dz\;\frac{\Gamma(-z)\Gamma(1-z)}{\Gamma(z+2)}(\kappa^2)^z\;\left(\frac{-q^2}{16\,\pi^2}\right)^{z-1} \qquad (6.14)$$

Then, in the notation of the previous section,

$$R_{\sigma\rho\mu}(k_1,k_2) = \frac{1}{2\pi i}\int_{C_0} dz\;\frac{\Gamma(-z)\;\Gamma(1-z)}{\Gamma(z+2)}\;\left(\frac{-\kappa^2}{16\pi^2}\right)^z\; R_{\sigma\rho\mu}(k_1,k_2,z)$$

where

$$R_{\sigma\rho\mu}(k_1,k_2,z) = 2\int d^4q\;\mathrm{Tr}[\,(\not{q}+\not{k}_1)\gamma_{\sigma}\,\not{q}\gamma_{\rho}\,(\not{q}-\not{k}_2)\gamma_{\mu}\gamma_5\,]\;\frac{1}{(q+k_1)^2}\;\frac{1}{(q-k_2)^2}\;(q^2)^{z-1} \qquad (6.15)$$

Note that $R_{\sigma\rho\mu}(k_1,k_2,0)$ is the expression given in eqn.(6.9) with $m = 0$. Using the algebra of Dirac-matrices, one can show that

$$\mathrm{Tr}[\,\gamma_{\alpha}\gamma_{\sigma}\gamma_{\beta}\gamma_{\rho}\gamma_{\delta}\gamma_{\mu}\gamma_5] = 4i[\,\eta_{\alpha\sigma}\varepsilon_{\beta\rho\delta\mu} - \eta_{\alpha\beta}\varepsilon_{\sigma\rho\delta\mu} + \eta_{\beta\sigma}\varepsilon_{\alpha\rho\delta\mu}$$

$$- \eta_{\alpha\rho}\varepsilon_{\beta\sigma\delta\mu} + \eta_{\rho\sigma}\varepsilon_{\beta\alpha\delta\mu} - \eta_{\rho\beta}\varepsilon_{\sigma\alpha\delta\mu} + \eta_{\delta\alpha}\varepsilon_{\rho\beta\sigma\mu} - \eta_{\delta\sigma}\varepsilon_{\rho\beta\alpha\mu}$$

$$+ \eta_{\delta\beta}\varepsilon_{\rho\sigma\alpha\mu} - \eta_{\delta\rho}\varepsilon_{\beta\sigma\alpha\mu}]$$

We substitute this in eqn.(6.15). We can combine the denominators using [41],

$$\frac{1}{(q-k_2)^2}\ \frac{1}{(q+k_1)^2}\ \frac{1}{(q^2)^{1-z}} = \frac{\Gamma(3-z)}{\Gamma(1-z)} \int_0^1 dx \int_0^{1-x} dy$$

$$\frac{(1-x-y)^{-z}}{[(q-k_2)^2 x + (q+k_1)^2 y + q^2(1-x-y)\ ]^{3-z}} \qquad (6.17)$$

and integrate over q using [42]

$$\int d^4q \quad \frac{[1, q^\alpha,\ q^\alpha q^\beta,\ q^\alpha q^\beta q^\delta]}{[q^2+2q.p+m^2\ ]^{3-z}} = \frac{i\ \pi^2}{(m^2-p^2)^{1-z}} \quad \frac{\Gamma(1-z)}{\Gamma(3-z)}$$

$$[1, -p^{\alpha},\ \{p^\alpha\ p^\beta - \frac{\eta^{\alpha\beta}}{2z}(m^2-p^2)\ \},$$

$$\{-p^\alpha\ p^\beta\ p^\delta - \frac{1}{2z}(\eta^{\alpha\beta}p^\delta + \eta^{\beta\delta}\ p^\alpha + \eta^{\alpha\delta}\ p^\beta)(m^2-p^2)\}] \qquad (6.18)$$

Now, the general structure for $R_{\sigma\rho\mu}$ consistent with the requirments of parity and Lorentz-invariance is [57],

$$R_{\sigma\rho\mu}\ (k_1,k_2) = A_1 k_1^\tau\ \varepsilon_{\tau\sigma\rho\mu} + A_2 k_2^\tau\ \varepsilon_{\tau\sigma\rho\mu} + A_3 k_{1\rho}\ k_1^\xi\ k_2^\tau\ \varepsilon_{\xi\tau\sigma\mu}$$

$$+ A_4 k_{2\rho}\ k_1^\xi\ k_2^\tau\ \varepsilon_{\xi\tau\sigma\mu} + A_5 k_{1\sigma}\ k_1^\xi\ k_2^\tau\ \varepsilon_{\xi\tau\rho\mu} + A_6 k_{2\sigma}\ k_1^\xi\ k_2^\tau\ \varepsilon_{\xi\tau\rho\mu} \qquad (6.19)$$

The requirement of Bose-symmetry $R_{\sigma\rho\mu}\ (k_1,k_2) = R_{\rho\sigma\mu}(k_2,k_1)$ implies that

$$\begin{aligned} A_1(k_1,k_2) &= -A_2(k_2,k_1) \\ A_3(k_1,k_2) &= -A_6(k_2,k_1) \\ A_4(k_1,k_2) &= -A_5(k_2,k_1) \end{aligned} \qquad (6.20)$$

Vector-current conservation $k_1^\sigma\ R_{\sigma\rho\mu} = k_2^\rho\ R_{\sigma\rho\mu} = 0$ gives the relations,

$$A_1 = k_1.k_2A_3 + k_2^2A_4$$
$$A_2 = k_1^2A_5 + k_1.k_2A_6 \qquad (6.21)$$

So, one needs to determine A_3 and A_4 only. From the above-mentioned steps, one finds:

$$A_{3,4}(k_1,k_2) = \frac{1}{2\pi i} \int_{C_0} dz \, \frac{\Gamma(-z)\,\Gamma(1-z)}{\Gamma(z+2)} \left(\frac{-\kappa^2}{16\pi^2}\right)^z A_{3,4}(k_1,k_2,z) \qquad (6.22)$$

where

$$A_3(k_1,k_2,z) = -16\pi^2 \int_0^1 dx \int_0^{1-x} dy \, \frac{xy(1-x-y)^{-z}}{[k_1^2y(1-y)+k_2^2x(1-x)+2xyk_1.k_2]^{1-z}} \qquad (6.23)$$

$$A_4(k_1,k_2,z) = -16\pi^2 \int_0^1 dx \int_0^{1-x} dy \, \frac{x(1-x)(1-x-y)^{-z}}{[k_1^2y(1-y)+k_2^2x(1-x)+2xyk_1.k_2]^{1-z}} \qquad (6.24)$$

Now $A_{3,4}(k_1,k_2,z)$ are regular at $z = 0$ and coincide with values mentioned in Ref. [57]. Evaluating the simple pole at $z = 0$ in eqn. (6.22), we find,

$$A_{3,4}(k_1,k_2) = A_{3,4}(k_1,k_2,0) + O(\kappa^2 \log\kappa) \qquad (6.25)$$

Now from eqn. (6.19),

$$-(k_1+k_2)^\mu R_{\sigma\rho\mu} = (A_1-A_2)k_1^\mu \; k_2^\tau \; \varepsilon_{\tau\sigma\rho\mu} \qquad (6.26)$$

Using the eqns. (6.20) to (6.26), one finds,

$$-(k_1+k_2)^\mu R_{\sigma\rho\mu} = 8\pi^2 \, k_1^\xi \; k_2^\tau \; \varepsilon_{\xi\tau\sigma\rho} + O(\kappa^2 \log\kappa) \qquad (6.27)$$

which to $O(\kappa)$ coincides with eqn. (6.12) when $m = 0$. Hence to this order, we have recovered the usual anomaly term.

6.3 Equations of Motion and the Anomaly Term

It is well known that the anomaly term in eqn.(6.13) can also be obtained by the equations of motion after defining the axial vector current in a gauge-invariant fashion [44,57,58]. Now we show that to $O(\kappa^0)$ we recover the same equation. The gravity-modified QED Lagrangian in our approximation is:

$$L = e^{-\kappa\chi} \left[\frac{i}{2}\left(\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - \partial_{\mu}\bar{\psi}\gamma^{\mu}\psi \right) - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e\bar{\psi}\gamma^{\mu}\psi A_{\mu} \right] + L_{grav} \tag{6.28}$$

The axial vector current j^5_{μ} and the pseudo-scalar current j^5 are given respectively by

$$\begin{aligned} j^5_{\mu} &= (\bar{\psi}\gamma_{\mu}\gamma_5\psi)e^{-\kappa\chi} \\ j^5 &= \bar{\psi}\gamma_5\psi\, e^{-\kappa\chi} \end{aligned} \tag{6.29}$$

Now we will regard these currents as the limits of non-local currents in which fields $\bar{\psi}$ and ψ are evaluated at separate points [44]:

$$j^A(x) = \lim_{\varepsilon \to 0} j^A(x, \varepsilon)$$

where

$$j^A(x, \varepsilon) = e^{-\kappa\chi(x)}\, \bar{\psi}\left(x + \frac{\varepsilon}{2}\right) \Gamma_A \psi\left(x - \frac{\varepsilon}{2}\right) \exp\left[-ie \int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} d\xi^{\mu} A_{\mu}(\xi)\right] \tag{6.30}$$

Here ε is a space-like vector and the integral in the exponential is over a space-like path. $\Gamma_A = \gamma_{\mu}\gamma_5$ for j^5_{μ} and $\Gamma_A = \gamma_5$ for j^5. j^A will be invariant under the gauge transformation,

$$\psi \rightarrow e^{+ie\,\Lambda(x)}\,\psi(x)$$
$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda(x) \tag{6.31}$$

Using the equations of motion,

$$i\,\partial_\mu(\bar{\psi}\gamma^\mu e^{-\kappa\chi}) + m\bar{\psi}\,e^{-\kappa\chi} - e\bar{\psi}\,\gamma_\mu\,A^\mu\,e^{-\kappa\chi} = 0$$
$$i\gamma^\mu\partial_\mu\,(\psi\,e^{-\kappa\chi}) - m\psi\,e^{-\kappa\chi} + e\gamma_\mu\,A^\mu\psi\,e^{-\kappa\chi} = 0 \tag{6.32}$$

We find,

$$\partial_\mu j^{5\mu}(x,\,\varepsilon) = -\kappa\partial_\mu\,\chi j^{5\mu}(x,\,\varepsilon) - iej^5_\mu(x,\,\varepsilon)\varepsilon_\lambda\,F^{\mu\lambda} + 2imj^5(x,\varepsilon) \tag{6.33}$$

The second term on the RHS of eqn.(6.33) is formally of $O(\varepsilon)$. But a careful evaluation, by calculating $(0\,|\,j^5_\mu(x,\varepsilon)|\,0)$ shows that it is $O(\varepsilon^{-1})$ and

$$ie(0\,|\,j^5_\mu(x,\,\varepsilon)\,\varepsilon_\lambda|\,0)F^{\mu\lambda} = \frac{e^2}{16\,\pi^2}\,\varepsilon_{\mu\lambda\xi\eta}\,F^{\xi\eta}\,F^{\mu\lambda} \tag{6.34}$$

So to $O(\kappa^0)$ we get the eqn.(6.13).

The gravitational contribution to anomaly has been evaluated by Delbourgo and Salam by considering the triangle diagram with two external graviton lines [62]. When this is included,

$$\partial_\mu j^{5\mu}(x,\,\varepsilon) = 2im\,j^5(x\varepsilon) + \frac{e^2}{16\,\pi^2}\,\varepsilon_{\mu\lambda\xi\eta}\,F^{\xi\eta}\,F^{\mu\lambda}$$
$$+ \frac{1}{76?\,\pi^2}\,\varepsilon_{\delta\lambda\mu\nu}\,R^{\delta\lambda}_{\ \ \rho\sigma}\,R^{\mu\nu\rho\sigma} \tag{6.35}$$

where $R_{\delta\lambda\rho\sigma}$ is the curvature tensor. $R_{\delta\lambda\rho\sigma} \sim O(\kappa)$.

Non-Abelian gauge theories

The axial anomaly for non-Abelian gauge theories can be derived using a point-splitting method for these theories [63]. Consider a Lagrangian invariant under local transformations of a non-Abelian group:

$$L = \bar{\psi}\,[i\gamma^{\mu}(\partial_{\mu} - ig\,A^{a}_{\mu}\ \tfrac{1}{2}\lambda^{a}) - m]\,\psi + L_{Y.M.} \tag{6.36}$$

Here A^{a}_{μ} are the gauge fields and $\frac{1}{2}\lambda^{a}$ are the matrix representation of the generators of the group. Then the axial-vector currents,

$$j^{5\mu} = \bar{\psi}\gamma^{\mu}\gamma_5\,\psi$$

$$\text{and}\quad j^{5\mu a} \equiv \bar{\psi}\gamma^{\mu}\gamma_5\,\tfrac{1}{2}\lambda_a\,\psi \tag{6.37}$$

would satisfy the anomalous Ward identities [63]

$$iD_{\mu}\,j^{5\mu} + 2mj^{5} = \frac{g^2}{32\pi^2}\,\varepsilon_{\lambda\mu\nu\rho}\,F^{\lambda\mu a}\,F^{\nu\rho\,a}$$

and

$$iD_{\mu}\,j^{5\ \mu a} + 2m\,j^{5a} = \frac{g^2}{64\pi^2}\,d_{abc}\,\varepsilon_{\lambda\mu\nu\rho}\,F^{\lambda\mu b}\,F^{\nu\rho c} \tag{6.38}$$

where j^5, j^{5a} are the corresponding pseudo-scalar currents, D_{μ} stands for gauge-covariant derivative and d_{abc} have the usual significance for the representation to which fermions belong. In the gravity-modified theory these relations would clearly hold to $O(\kappa^{0})$, as in the case of QED.

CHAPTER VII

CONCLUDING REMARKS

A unified theory for particle interactions has to include quantized gravity, in the final analysis. The problems associated with the quantization of gravity are not yet fully resolved [64]. We have followed a 'particle physicist's approach' in which gravity is treated as a gauge theory and quantized in a manner identical to non-Abelian gauge theories [64]. The gravitational Lagrangian is intrinsically non-polynomial. This, along with the complexities of a gauge theory makes quantum gravity difficult to handle. However, we have shown that it is sufficient to work in an approximation to full tensor gravity, which greatly simplifies calculations, to regularize the ultraviolet infinities in a general quantum field theory when couplings with the gravitational field are incorporated in it. The regularizing mechanism extends to non-renormalizable situations also. The inverse of the gravitational coupling constant, κ, plays the role of a cutoff. We have studied the specific details of the regularization in various Lagrangians of physical interest namely theories with spontaneous symmetry breaking, non-Abelian gauge theories and theories involving axial vector currents.

Now, any regularization scheme should respect Poincare invariance and the various global and local internal symmetries associated with the theory. With gravity treated as above, the former clearly holds when we consider terms to $O(\kappa^{0})$. For the σ-model, we have shown that the vertex and self-energy corrections are consistent with the global symmetry of the theory, to $O(\kappa^{0})$. Goldstone theorem and PCAC are also verified in the same order. A special feature of this model is the regularization of the bubble diagrams.

The situation is slightly different for theories with local symmetries. For the non-Abelian gauge theories, we find that the Ward-identities hold to $O(\log \kappa)$ only. Here we have to use extra considerations, like a manifestly gauge covariant formalism to obtain gauge invariant results. For the axial vector anomaly, we get the same answer as the standard one to $O(\kappa^{0})$.

We have considered only the regularizing role of gravity when it is coupled to other fields. The gravitational corrections to physical amplitudes have not been discussed. Here the self-interactions of gravity can no longer be ignored.

Of late, there is renewed interest in the renormalizability aspects of quantum gravity. G.'t Hooft and M. Veltman have shown that pure gravity is renormalizable

upto one-loop order [65]. When matter fields are included, this conclusion is no longer valid. It has been explicitly demonstrated that matter fields destroy renormalizability in this order [66]. This may well be due to the fact that ordinary perturbation theory provides an inadquate framework for discussion of quantum gravity. The application of non-polynomial methods to the self-interactions of gravity would clarify the situation. The computational difficulties here are, however, formidable.

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